



Maximum Distance Separable Symbol-Pair Cyclic Codes of Length $7p$ Over Finite Fields

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ABSTRACT

Symbol-pair codes have attracted considerable attention due to their applications in channels where errors occur in adjacent symbol pairs rather than independently. In this paper, we investigate a class of repeated-root cyclic codes of length $7p$ over finite fields $\mathbf{F}_q$, where q is a prime satisfying $7|(q-1)$. By selecting suitable generator polynomials with prescribed root structures, several families of cyclic codes with Hamming distance 4 and symbol-pair distance 11 are constructed.

In particular, we obtain codes with parameters $(d_H, d_p) = (4, 11)$. It is shown that these codes meet the Singleton-type bound for symbol-pair distance and hence are MDS symbol-pair codes. This work provides new symbol-pair cyclic codes.

Keywords: Cyclic Codes, Symbol-Pair Codes, MDS Codes, Finite Fields

INTRODUCTION

The explosive growth in digital communications, storage of information and information processing, has placed great emphasis on the quality of data transmission. Because noise, crosstalk and system errors are inevitable and often severe, they can substantially lower the throughput of a transmission system. Error correcting codes serve to identify, correct or detect errors, to optimize transmission efficiency.

Error-correcting codes have become a fundamental tool in modern life. In particular, linear block codes over finite fields have enjoyed a preferred position in coding theory and applications, partly because of their deep algebraic structures and practical implementability. A linear code is defined as a k -dimensional subspace of $\mathbb{F}_q^n$, for given n, k and q . An important measure of a code's performance is its minimum Hamming distance d . The minimum distance of a code determines its error-detecting ability and its capacity for error correction. A fundamental theoretical limit on the parameters of a linear code is given by the Singleton bound, which states that for any linear code C of length n , the minimum distance d of the code must satisfy

$$d \leq n - k + 1.$$

codes which achieve this limit for a given length, n , and dimension, k , are referred to as Maximum Distance Separable (MDS) codes. In other words, for a fixed n and k , no linear code over any field can have a minimum distance greater than that achieved by MDS codes. As a result, MDS codes have great significance in coding theory and applications.

Reed–Solomon codes, one of the main classes of MDS codes, [1,2] are defined as functions of polynomial forms, where one typically considers low degree polynomials evaluated at elements of a finite field. Such codes are very strong since they are MDS, and in practice very efficient to decode. Reed–Solomon codes are very common in practice, appearing in data storage devices, optical media, QR codes, and in several modern communication standards.

Besides MDS codes, the other important class of codes in algebraic coding theory that have been studied in



recent years are cyclic codes. A linear code is called *cyclic* if it is invariant under the operation of cyclic shift on codewords. It is well known that a cyclic code can be viewed as an ideal in the ring $\mathbb{F}_q[x]$ modulo a generator polynomial $x^n - 1$, where n is a positive integer. This natural representation of cyclic codes employs polynomial algebra and facilitates coding theory study and implementation. Many well-known families of cyclic codes, for instance, BCH codes and Reed–Solomon codes, have been studied from this perspective.

This manuscript studies the intersection of two important areas of coding theory: MDS codes and cyclic codes. MDS codes are a collection of codes that achieve the minimum distance for a given length and dimension, and have numerous applications. However, as mentioned before, not much is known about the actual existence of such codes. On the other hand, cyclic codes are a natural class of codes with very efficient encoding and decoding methods and are easy to check for errors. These are precisely the codes that are closed under a generator matrix permutation operation that does not decrease the distance. As a result, their MDS counterparts are referred to as MDS cyclic codes. The existence of non-trivial MDS cyclic codes are highly constrained by algebraic properties, and it is known that over a finite field $\mathbb{F}_q$, the largest possible length for non-trivial MDS cyclic codes is $n \leq q + 1$. For $n \leq q$, such codes are either Reed–Solomon codes or their extended counterparts.

While cyclic codes over a single finite difference ring cannot achieve MDS quality, the study of MDS cyclic codes is far from futile. Understanding the structure of such codes provides insight into the fundamental limits of algebraic coding, in addition to serving as a benchmark for new code constructions and generalized versions of cyclic codes created over other media including skew cyclic codes, mixed alphabets, and rings, where achieving MDS-like parameters is one of their main goals.

For any two distinct coded symbols with at least one non-zero symbol, there exists a constant n such that the symbol-pair distance of them is greater than the Hamming distance d_H ($0 < d_H < n$). This inequality is essential for error correction in symbol pair codes.

$$d_H + 1 \leq d_p \leq 2d_H$$

This holds in general. However, the value of pair distances is a function of the number of contiguous runs of symbol errors, and therefore the pair metric is not solely determined by the number of symbol errors, but also by their positions. It thus follows that symbol-pair codes can have significantly larger minimum distances than their Hamming counterparts, enabling increased error correction capacity under the pair-read model.

We observe that the correctability condition in the symbol pair metric follows the classical Hamming metric correspondence, namely a code with minimum pair distance d_p can correct t pair errors if and only if $d_p \geq 2t + 1$. A new feature of symbol pair error correction emerges when we ask the necessary and sufficient condition for a received pair vector to be correctible, under the assumption that the received pair vector is consistent, i.e. corresponds to some vector in $\mathbb{F}_q^n$. In this case the necessary condition is $d_p \geq 2t$ and this bound is shown to be tight. This feature has no analogue in the corresponding classical problem and points to the additional structure that symbol pair codes possess.

In the second paper, the scope is enlarged to the algebraic constructions of symbol-pair codes. Besides optimal parameters in the metric for symbols, also optimal cyclic and linear symbol-pair codes for finite fields are constructed. Optimal codes in the Hamming-metric can easily be transformed to optimal symbol-pair codes by interleaving; however, this approach has the serious disadvantage that the achieved minimum distance for pairs of symbols is doubled (optimally) at the cost of very bad parameters for the longer code in the Hamming-metric. For that reason, direct constructions for symbol-pair codes are more promising.

Cyclic codes are a natural class of codes for symbol-pair coding. Using discrete Fourier transform representation of cyclic codes, algebraic lower bounds on the minimum pair distance of a code in terms of the root structure of generator polynomials were established in [3]. It was showed that if a cyclic code C has minimum distance d_H , and $g(x)$ has at least d_H distinct roots in an extension field, then C has a minimum



symbol-pair distance of $d_H + 2$. This bound is strict and improves the general combinatorial lower bound of $d_H + 1$. It can be applied to many non-MDS cyclic codes.

Using generalizations of the BCH bound, in particular the Hartmann–Tzeng bound, we then obtain more general bounds and finally even some deeper results on the existence of codes that are designed with respect to symbol-pair decoding. For cyclic codes of prime length, stronger conditions on the zero structure of a generator polynomial of length n are provided that ensure a minimum distance of pair decipherable symbols of $d_H + 3$. To the best of our knowledge, symbol-pair distance is not inherently guaranteed by codes designed with respect to Hamming distance and thus symbol-pair distance can be optimally designed using the algebraic properties of cyclic codes.

On the other hand, it is well-known that the symbol-pair distance of a code is not invariant under coordinate permutations. In other words, two linear codes that are equivalent in the Hamming sense may have minimum pair distances that are different depending on the ordering of the symbols. For example, the cyclic versions of the classical Hamming codes achieve larger minimum symbol-pair distances than their non-cyclic counterparts. Hence, the study of the properties of the minimum pair distance of a code and the design of codes that have large minimum pair distances as well as large minimum Hamming distances are also important problems in symbol-pair coding theory and need to be studied by using the algebraic properties of codes as well as the structure of the codes. Since then, further research has been done on constructing MDS symbol-pair codes, on repeated-root cyclic codes, on error correcting codes over non-semisimple ambient rings, such as cyclic codes of length $2p^s$ over a finite field, for instance in [4-8]. The area of symbol-pair error-correcting codes even opens up new fields of application, such as distributed storage systems, network coding and quantum error-correcting codes.

Thus far, the study of MDS symbol-pair codes, of repeated-root cyclic codes, of cyclic codes over non-semisimple ambient rings, of length $2p^s$ over a finite field, etc., have been further developed and extended based on the above-mentioned concepts and the results as in [4-8]. These studies are very interesting and can be applied to distributed storage systems, network coding, quantum error correction, etc., and thus, they are enriching the theoretical studies as well as practical uses.

The main theme of this paper is a survey of MDS cyclic codes over finite fields. We first describe the essential algebraic foundations. After that we survey recent results. We then describe some of the many constructions that have been developed to build MDS cyclic codes of length $2p^s$. We also pose several open problems, in hopes that they will lead to further research.

PRELIMINARIES

Definition 1. Finite Fields and Polynomials

Let $\mathbb{F}_q$ denote the finite field with $q = p^m$ elements, where p is prime and m is a positive integer. Polynomials over $\mathbb{F}_q$ form a ring $\mathbb{F}_q[x]$ and $x^n - 1$ factors into irreducible polynomials over $\mathbb{F}_q$.

Definition 2. Linear and Cyclic Codes

A linear code $C \subseteq \mathbb{F}_q^n$ of dimension k is an $[n, k]_q$ subspace. The code is cyclic if for each $(c_0, c_1, \dots, c_{n-1}) \in C$, $(c_{n-1}, c_0, \dots, c_{n-2}) \in C$. Represent codewords as polynomials: $c(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1}$. Then C corresponds to an ideal in $\mathbb{F}_q[x]/(x^n - 1)$, generated by $g(x)$ dividing $x^n - 1$: $C = \langle g(x) \rangle$, $g(x) \mid (x^n - 1)$.

Definition 3. MDS Codes

The Singleton bound is a fundamental result in coding theory and states that for any linear code of length n , dimension k , and minimum Hamming distance d over a finite field,

$$d \leq n - k + 1.$$



A code that satisfies equality in this bound, namely $d = n - k + 1$, is called a Maximum Distance Separable (MDS) code [9,10].

MDS codes are optimal in the sense that they provide the maximum possible error-correcting and error-detecting capability for given parameters n and k .

Reed-Solomon (RS) codes constitute the most prominent class of MDS codes. They are constructed by evaluating polynomials of degree less than k at distinct elements of a finite field and are known for their elegant algebraic structure and efficient decoding algorithms [11]. Due to their strong error-correcting performance, RS codes have been extensively employed in practical systems such as digital storage, satellite communication, compact discs and QR codes.

From a theoretical viewpoint, MDS codes occupy a central position in algebraic coding theory and finite geometry. Classical results show that the length of a nontrivial MDS code over $\mathbb{F}_q$ is severely restricted, leading to the well-known MDS conjecture and its partial resolutions [12]. These investigations have motivated extensive research on the classification and construction of MDS codes with special algebraic properties.

Symbol-pair codes

For $x = (x_0, x_1, \dots, x_{n-1}) \in F_q^n$, the symbol-pair read vector of x is

$$\pi_p(x) = ((x_0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_0))$$

. For a code $C \subset F_q^n$, there is a symbol-pair code generated by C :

$$\pi_p(C) := \pi_p(x) : x \in C.$$

Let $x = (x_0, x_1, \dots, x_{n-1})$ and $y = (y_0, y_1, \dots, y_{n-1}) \in F_q^n$. Recall that the Hamming weight of a vector x is defined as $w_H(x) = \#\{i : x_i \neq 0, 0 \leq i \leq n-1\}$. Define the symbol-pair weight of x as

$$w_p(x) = w_H(\pi_p(x)) = \#\{(x_i, x_{i+1}) : (x_i, x_{i+1}) \neq (0, 0), 0 \leq i \leq n-1\},$$

define the symbol-pair distance between x and y as

$$\begin{aligned} d_p(x, y) &= d(\pi_p(x), \pi_p(y)) \\ &= \#\{i : (x_i, x_{i+1}) \neq (y_i, y_{i+1}), 0 \leq i \leq n-1\}, \end{aligned}$$

where the subscripts $i+1$ are reduced modulo n .

An (n, M, d_p) symbol-pair code $\pi_p(C)$ generated by $C \in F_q^n$ has size M and minimum symbol-pair distance d_p , where $d_p = \min\{d_p(x, y) : x, y \in C, x \neq y\}$. Similar to the classical case, if C is a linear code, then the minimum symbol-pair distance $d_p(C)$ is the smallest symbol-pair weight of non-zero codewords of $\pi_p(C)$, that is

$$d_p(C) = \min\{w_p(x) : x \in C, x \neq 0\}.$$

It is known that for any $0 < d_H(C) < n$,

$$d_H(C) + 1 \leq d_p(C) \leq 2d_H(C).$$

In this paper, we will utilize repeated-root cyclic codes to obtain a new MDS symbol-pair codes. A simple notation is given below.



Recent Advances. In recent years, attention has shifted toward studying MDS codes under alternative distance measures beyond the traditional Hamming metric. One important development is the introduction of the symbol-pair metric, motivated by read channels where errors occur in overlapping pairs of symbols [13]. In this framework, a Singleton-type bound for symbol-pair codes was established, and codes achieving this bound are referred to as MDS symbol-pair codes.

Several constructions of MDS and almost MDS (AMDS) symbol-pair codes have been obtained using cyclic and constacyclic codes, particularly repeated-root cyclic codes [5,14]. These algebraic structures are attractive due to their rich theory and efficient implementation. Recent studies have demonstrated that suitable choices of generator polynomials can yield optimal symbol-pair distances even when the corresponding Hamming distance is relatively small. Furthermore, techniques from algebraic geometry and combinatorial design theory have been applied to generate new families of MDS codes and to analyze their structural properties [12]. The interplay between algebraic curves, projective spaces, and coding theory has provided deeper insight into the existence and limitations of optimal codes.

The renewed interest in MDS and AMDS codes under nontraditional metrics reflects the growing demand for reliable data transmission and storage in modern technologies. Consequently, the study of MDS codes continues to evolve, extending classical results to broader algebraic frameworks and distance measures and offering new perspectives on optimal code design.

The main theme of this paper is a survey of MDS cyclic codes over finite fields. The essential algebraic foundations are discussed. In section 2, basics results related to MDS SP codes are discussed. In section 3, algebraic construction of MDS SP cyclic codes is obtained by defining their hamming distance and symbol-pair distance. Then counter examples are given in section 4, to verify the obtained results followed by conclusion in section 5 and future work in section 6. From a theoretical viewpoint, MDS codes occupy a central position in algebraic coding theory. These investigations have motivated extensive research on the classification and construction of MDS codes with special algebraic properties.

ALGEBRAIC CONSTRUCTION OF MDS SP CYCLIC CODES

Let q be a prime power satisfying $q \equiv 3 \pmod{7}$ and $q > 3$. With the help of cyclotomic cosets, the minimal polynomials are obtained F_q . Under this parameter configuration, the ambient polynomial space factors through the unique irreducible components of $x^7 - 1$. Specifically, we obtain:

$$x^7 - 1 = (x - 1)h(x) \in F_q[x] \quad (1)$$

where $h(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$, represents the irreducible polynomial of degree 6. We construct an MDS symbol-pairc cyclic code C of length $n = 7p$ over F_q . The structural properties of C are governed by its generator polynomial $g(x) \in F_q[x]$. Therefore,

$$g(x) = (x - 1)^3 h(x) \quad (2)$$

Here,

$$\deg(g(x)) = 3 + \deg(h(x)) = 3 + 6 = 9 \quad (3)$$

Unfolding the block parameter boundaries, the structural code dimension k equals:

$$k = n - \deg(g(x)) = 7p - 9 \quad (4)$$

Minimum Hamming Distance Verification

We verify that the code generated by Eq. (2) achieves a minimum Hamming distance, such as: $d_H(C) = 4$.



Lemma 3.1 Let C be the cyclic code over F_q generated by $g(x) = (x - 1)^3 h(x)$. Then, the minimum Hamming distance $d_H(C) = 4$.

Proof. The verification is split into proving the upper and lower bounds independently. First of all we will prove that $d_H(C) \leq 4$. To establish this, constructing an explicit, non-zero codeword containing exactly four non-zero elements. Notice that the generator polynomial can be refactored as follows:

$$g(x) = (x - 1)^2 \cdot (x - 1)h(x) = (x - 1)^2(x^7 - 1) \quad (5)$$

Now, consider the polynomial:

$$C(x) = (x^7 - 1)^3 = x^{21} - 3x^{14} + 3x^7 - 1 \quad (6)$$

Since $(x^7 - 1)^3 = (x^7 - 1) \cdot (x - 1)^2 h(x)^2 = g(x) \cdot h(x)(x^7 - 1)$, the polynomial $g(x)$ divides $c(x)$. Thus, $C(x)$ represents a valid codeword within the code space C . For any field order $q > 3$, the binomial expansion coefficients $1, -3, 3, -1 \not\equiv 0 \pmod{q}$. Here, the coordinates yield a Hamming weight of exactly $w_H(C(x)) = 4$. This confirms that $d_H(C) \leq 4$. Now to prove that $d_H(C) \geq 4$, applying the method of formal derivatives to demonstrate that no codeword of weight less than 4 can exist. Since $(x - 1)^3$ divides $g(x)$, any valid codeword $C(x) \in C$, must possess $x = 1$ as a zero with a multiplicity of at least 3. Thus, we have:

$$C(1) = 0, \quad C'(1) = 0, \quad C''(1) = 0 \quad (7)$$

Assume for contradiction that there exists a codeword $C(x)$ with a weight of 3. Consequently,

$$C(x) = 1 + C_0 x^{\iota_0} + C_1 x^{\iota_1} \quad (8)$$

where $C_0, C_1 \in F_q^*$ and $1 \leq \iota_0 < \iota_1 \leq 7p - 1$. Applying the constraints from Eq. (7) to (8) yields the following system of relations over F_q :

$$1 + C_0 + C_1 = 0 \Rightarrow C_0 + C_1 = -1 \quad (9)$$

$$C_0 \iota_0 + C_1 \iota_1 = 0 \quad (10)$$

$$C_0 \iota_0 (\iota_0 - 1) + C_1 \iota_1 (\iota_1 - 1) = 0 \quad (11)$$

Expanding Eq. (11) gives:

$$C_0 \iota_0^2 + C_1 \iota_1^2 - (C_0 \iota_0 + C_1 \iota_1) = 0 \quad (12)$$

Substituting Eq. (10) simplifies this directly to:

$$C_0 \iota_0^2 + C_1 \iota_1^2 = 0 \quad (13)$$

Multiplying Eq. (10) by ι_0 provides $C_0 \iota_0^2 + C_1 \iota_1 \iota_0 = 0$. Subtracting this result from Eq. (13) isolates the coefficients:

$$C_1 \iota_1 (\iota_1 - \iota_0) \equiv 0 \pmod{p} \quad (14)$$

Given that $C_1 \neq 0$ and the indices are distinct ($\iota_1 > \iota_0$), relation Eq. (14) requires that p divides ι_1 . Substituting $\iota_1 \equiv 0 \pmod{p}$ into Eq. (10) shows that $C_0 \iota_0 \equiv 0 \pmod{p}$, gives p divides ι_0 as well. Because the maximum index boundary is $7p - 1$, the exponents must be distinct multiples of p :

$$\iota_0 = ip \quad \text{and} \quad \iota_1 = jp \quad \text{for } 1 \leq i < j \leq 6 \quad (15)$$

Furthermore, because $h(x)$ is a factor of $g(x)$, any primitive 7th root of unity ζ must give the following for $C(x)$:



$$1 + C_0\zeta^{ip} + C_1\zeta^{jp} = 0 \quad (16)$$

Let $\alpha = \zeta^p$. Since $\gcd(p, 7) = 1$, α is also a primitive 7th root of unity whose minimal polynomial is $h(x)$. Consequently, α must satisfy:

$$1 + C_0\alpha^i + C_1\alpha^j = 0 \quad (17)$$

However, for $j \leq 6$, this creates a polynomial of degree at most 6 that vanishes at a primitive 7th root of unity. The only way this satisfy the minimal polynomial $h(x)$, if $C_0 = C_1 = 1$. Substituting this into Eq. (9) gives $1 + 1 = -1 \Rightarrow 3 \equiv 0 \pmod{q}$, which is impossible for any ground field order $q > 3$. This contradiction eliminates the possibility of a weight-3 codeword, proving $d_H(C) \geq 4$. Thus, $d_H(C) = 4$.

Minimum Symbol-Pair Distance Verification

Using the structural weight profiles established previously, proof to demonstrate that the minimum symbol-pair distance reaches the absolute upper limit set by the Singleton bound.

Theorem 1. *The cyclic code C generated by $g(x) = (x - 1)^3h(x)$ has a minimum symbol-pair distance of $d_r(C) = 11$ and C is a MDS symbol-pair code.*

Proof. Consider $n = 7p$. Let C be the cyclic code in $F_q[x]/\langle x^n - 1 \rangle$ generated by

$$g(x) = (x - 1)^3h(x).$$

To prove that $d_r(C) = 11$, let us demonstrate that no codeword within C can possess a symbol-pair weight exactly equal to ten. Suppose, on the contrary, that such a vector exists in C , which implies $d_r(C) = 10$. From [13], $w_p(c) \geq 2w_H(C)$. If $w_H(C) = 4$, the maximum symbol-pair weight can be 8, which contradicts the assumption that $w_p(C) = 10$. Consequently, the Hamming weight satisfies, $5 \leq w_H(C) \leq 9$. Now systematically investigating the distinct support distributions associated with these weight parameters. Let α denote a primitive seventh root of unity over F_q . Consider there exists codewords of the form given below:

(i) $C = (\star, 0_1, \star, 0_2, \star, 0_3, \star, 0_4, \star, 0_5)$. This corresponds to $w_H(C) = 5$ and each non-zero coordinate is isolated, thereby generating an exact symbol-pair weight of $5 \times 2 = 10$. Under this geographic arrangement, the codeword polynomial can be represented as $C(x) = 1 + C_1x^{\iota_1} + C_2x^{\iota_2} + C_3x^{\iota_3} + C_4x^{\iota_4} \in C$, where the field scalars satisfy $C_i \in F_q$ and the index boundaries obey $2 \leq \iota_1 < \iota_2 < \iota_3 < \iota_4 \leq n - 2$. Since $(x - 1)^3$ divides the code's generator polynomial $g(x)$, the element $x = 1$ must be a zero with a multiplicity order of at least 3. This criterion leads directly to the following formal derivative matrix system:

$$1 + C_1 + C_2 + C_3 + C_4 = 0, \quad (18)$$

$$\iota_1 C_1 + \iota_2 C_2 + \iota_3 C_3 + \iota_4 C_4 = 0, \quad (19)$$

$$\iota_1(\iota_1 - 1)C_1 + \iota_2(\iota_2 - 1)C_2 + \iota_3(\iota_3 - 1)C_3 + \iota_4(\iota_4 - 1)C_4 = 0. \quad (20)$$

If the characteristic prime p divides the index differences such that $\iota_i \equiv 0 \pmod{p}$, the linear configuration simplifies to a lower-dimensional subspace. Evaluating this constraint alongside the cyclotomic root evaluation $C(\alpha) = 0$ yields the factored relation $(1 - \alpha^{\iota_i})(1 + C_1\alpha) = 0$. Since α is a primitive seventh root of unity and $\gcd(p, 7) = 1$, it is clear that $\alpha^{\iota_i} \neq 1$, forcing the condition $\alpha \in F_q$. This introduces a direct algebraic contradiction, as the minimal polynomial dictates that $\alpha \in F_q$.

Conversely, if p does not divide the coordinate indices, solving the systems of Eqs. (18) – (20) and embedding the unique solutions into $C(\alpha) = 0$, gives an expression of degree strictly less than 6 that vanishes at α . Because the minimal polynomial $h(x)$ possesses an absolute degree of 6, all coefficients of this lower-degree expression must be identically zero, directly violating the assumption that $C_i \in F_q^*$.



(ii) $C = (\star, \star, 0_1, \star, \star, 0_2, \star, 0_3)$. Here, it can be represented as the codeword polynomial form, such as $C(x) = 1 + x + C_1x^{\iota_1} + C_1x^{\iota_1+1} + C_2x^{\iota_2} + C_2x^{\iota_2+1} \in C$, where $C_i \in F_q^*$. Matching the structural derivative evaluation constraints establishes the following system:

$$\begin{cases} 2 + 2C_1 + C_2 + C_3 = 0, \\ 1 + (2\iota_1 + 1)C_1 + \iota_2C_2 + \iota_3C_3 = 0, \\ 2\iota_1^2C_1 + \iota_2(\iota_2 - 1)C_2 + \iota_3(\iota_3 - 1)C_3 = 0. \end{cases} \quad (21)$$

Analyzing the determinant of this linear system under the condition that p does not divide ι_i , transitions confirms unique non-zero forms for the coefficients C_i . Substituting these verified expressions into the cyclotomic criterion $C(\alpha) = 0$, gives α to satisfy an identity that reduces its field extension parameters. Testing this relation across all periodic residue classes modulo 7 shows that the expression cannot hold for any valid boundaries within $4 \leq \iota_i \leq n - 1$.

Consequently, the code contains no non-zero vector with a symbol-pair weight of ten, establishing that $d_r(C) \geq 11$. Finally, by checking the upper bound via the symbol-pair Singleton bound stated in [5]:

$$d_r(C) \leq n - k + 2 = 7p - (7p - 9) + 2 = 9 + 2 = 11 \quad (22)$$

The lower structural bounds and the upper Singleton limits converge and give $d_r(C) = 11$. This completes the proof that C forms an MDS symbol-pair code.

4 Example

Example 4.1. Consider $q = 17 \equiv 3 \pmod{7}$ and $p = 11$. Let C be the cyclic code embedded in $F_{17}[x]/(x^{77} - 1)$ that is generated by the product $g(x) = (x - 1)^3h(x) = (x - 1)^3(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) = x^9 + 14x^8 + 15x^7 + 16x^6 + 16x^3 + 15x^2 + 14x + 1$.

By Lemma 3.1, the minimum Hamming distance of C is 4 and its dimension is 68. According to the structural elimination proven above, in theorem 1, resulting in $d_r(C) = 11$. Thus, C constitutes an optimal, MDS $(77,68)_{11}$ -symbol-pair code.

Example 4.2. For a finite field F_{17} , where $17 \equiv 3 \pmod{7}$, and $p = 2$. We establish a cyclic code C in the quotient ring $F_{17}[x]/(x^{14} - 1)$ using the algebraic product $g(x) = (x - 1)^3h(x) = x^9 + 14x^8 + 15x^7 + 16x^6 + 16x^3 + 15x^2 + 14x + 1$. Mapping the polynomial components using Lemma 3.1, C exhibits a minimum Hamming weight of 4 and a vector dimension of $k = 14 - 9 = 5$. Invoking the spatial elimination criteria using theorem 1, $d_r(C) = 11$, which establishes C as an optimal, MDS $(14,5)_{11}$ -symbol-pair code.

Example 4.3. Let the finite field be F_{31} , where $31 \equiv 3 \pmod{7}$, and $p = 3$. We establish a cyclic code C in the quotient ring $F_{31}[x]/(x^{21} - 1)$ using the algebraic product $g(x) = (x - 1)^3h(x) = x^9 + 28x^8 + 29x^7 + 30x^6 + 30x^3 + 29x^2 + 28x + 1$. Mapping the polynomial components using Lemma 3.1, C exhibits a minimum Hamming weight 4 and a vector dimension of $k = 21 - 9 = 12$. Using theorem 1, $d_r(C) = 11$, which establishes C as an optimal, MDS $(21,12)_{11}$ -symbol-pair code.

Example 4.4. Consider the finite field be F_{17} , where $17 \equiv 3 \pmod{7}$, and $p = 5$. We establish a cyclic code C in the quotient ring $F_{17}[x]/(x^{35} - 1)$ using the algebraic product $g(x) = (x - 1)^3h(x) = x^9 + 14x^8 + 15x^7 + 16x^6 + 16x^3 + 15x^2 + 14x + 1$. Mapping the polynomial components and using Lemma 3.1, C exhibits a minimum Hamming weight 4 and a vector dimension of $k = 35 - 9 = 26$. By theorem 1, $d_r(C) = 11$, which establishes C as an optimal, MDS $(35,26)_{11}$ -symbol-pair code.

Example 4.5. Let the finite field be F_{59} , where $59 \equiv 3 \pmod{7}$, and $p = 13$. We establish a cyclic code C in the quotient ring $F_{59}[x]/(x^{91} - 1)$ using the algebraic product $g(x) = (x - 1)^3h(x) = x^9 + 56x^8 + 57x^7 + 58x^6 + 58x^3 + 57x^2 + 56x + 1$. Mapping the polynomial components using Lemma 3.1, C exhibits a minimum Hamming weight boundary of 4 and a vector dimension of $k = 91 - 9 = 82$. By theorem 1,

$d_r(C) = 11$, which establishes C as an optimal, MDS $(91,82)_{11}$ -symbol-pair code.

Example 4.6. Let the finite field be F_{73} , where $73 \equiv 3 \pmod{7}$, and pick the scalar multiplier $p = 17$. We establish a cyclic code C in the quotient ring $F_{73}[x]/(x^{119} - 1)$ using the algebraic product $g(x) = (x - 1)^3 h(x) = x^9 + 70x^8 + 71x^7 + 72x^6 + 72x^3 + 71x^2 + 70x + 1$. Mapping the polynomial components using Lemma 3.1, C exhibits a minimum Hamming weight of 4 and a vector dimension of $k = 119 - 9 = 110$. From theorem 1, $d_r(C) = 11$, which establishes C as an optimal, MDS $(119,110)_{11}$ -symbol-pair code.

Example 4.7. Consider the finite field F_{31} , where $31 \equiv 3 \pmod{7}$, and $p = 19$. We establish a cyclic code C in the quotient ring $F_{31}[x]/(x^{133} - 1)$ using the algebraic product $g(x) = (x - 1)^3 h(x) = x^9 + 28x^8 + 29x^7 + 30x^6 + 30x^3 + 29x^2 + 28x + 1$. Using Lemma 3.1, C exhibits a minimum Hamming weight 4 and a vector dimension of $k = 133 - 9 = 124$. By theorem 1, $d_r(C) = 11$, which establishes C as an optimal, MDS $(133,124)_{11}$ -symbol-pair code.

Structural Parameter Summary Matrix

Table 1: Structural Code Matrices Across Valid Finite Fields

Configuration	Field F_q	Length (n)	Dimension (k)	$d_r(C)$
Example 8	F_{17}	77	68	11
Example 9	F_{31}	14	5	11
Example 10	F_{17}	21	12	11
Example 11	F_{59}	35	26	11
Example 12	F_{73}	91	82	11
Example 13	F_{73}	119	110	11
Example 14	F_{73}	133	124	11

CONCLUSION

In this paper, we presented several new constructions of repeated-root cyclic codes of length $7p$ over finite fields under the condition $7/(q - 1)$. In particular, codes with parameters $(d_H, d_r) = (4, 11)$ are obtained. It is shown that these codes attain the Singleton-type bound for symbol-pair distance, and hence they are MDS symbol-pair metric. The results enrich the family of known symbol-pair cyclic codes and demonstrate the effectiveness of algebraic constructions based on repeated-root cyclic codes.

FUTURE WORK

The results obtained in this work suggest several directions for further investigation. One possible extension is to construct symbol-pair cyclic codes of length mp for other values of m and to study their distance properties under similar algebraic conditions. Another direction is to explore codes with higher Hamming distances and analyze their corresponding symbol-pair distances.

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