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Hα-Compact and Hα-Lindelof Topological Spaces

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ABSTRACT

Most references on topological spaces seem to define a compact space in terms of open coverings of a space having finite sub coverings. Formally, we say that a collection of open sets $\{o_\lambda : \lambda \in \Lambda\}$, indexed by some set Λ, is an open covering of a topological space (Z,T) if $Z \subseteq \cup_{\lambda\in\Lambda} o_\lambda$. We define Z to be compact if there exist finitely many open sets $o_{\lambda_1}, o_{\lambda_2}, ..., o_{\lambda_n}$ in the open covering such that $Z \subseteq \cup_{j=1}^{n} O_{\lambda_j}$. There exist several equivalent definitions of a compact space, which we briefly mention here. A topological space Z is compact if and only if Z has the finite intersection property: if $\{o_\lambda : \lambda \in \Lambda\}$ is a collection of a closed set indexed by a set Λ such that, for any finite subset $\Lambda^* \subseteq \Lambda$, $\cap_{\lambda\in\Lambda^*} O_\lambda \neq \phi$. then $\cap_{\lambda\in\Lambda} O_\lambda \neq \phi$. Another equivalent definition is in terms of nets. A topological space Z is compact if and only if every net $(z_\lambda : \lambda \in \Lambda)$ in Z has a convergent subnet. In 2022, Beyda S. Abdullah, Sabih W. Askandar and Ruqayah N. Balo introduced a new kind of open set, which is called an $h\alpha$-open set. They also studied the relationship between $h\alpha$-open sets and a variety of other open sets as h-open sets, open sets, semi-open sets, and α-open sets. Furthermore, they introduced the concepts of $h\alpha$-continuous mappings, $h\alpha$-open mappings, $h\alpha$-irresolute mappings, and $h\alpha$-totally continuous mappings. We will extend the concept of compactness via $h\alpha$-open sets by introducing $h\alpha$-compact spaces in topological spaces and will investigate their characterizations by making use of generalized mappings, including $h\alpha$-continuous and $h\alpha$-irresolute functions. The objective of this paper is to introduce the new concepts called $h\alpha$-compact space, $h\alpha$-Lindelöf space, countably $h\alpha$-compact space, almost $h\alpha$-compact space, and mildly $h\alpha$-compact space in topological spaces and investigate fundamental properties and characterizations of these new notions of spaces.

Keywords and Phrases: Topological space, $h\alpha$-open set, $h\alpha$-closed set, $h\alpha$-compact space, $h\alpha$-Lindelöf space, countably $h\alpha$-compact space, almost $h\alpha$-compact space, mildly $h\alpha$-compact space.

1. INTRODUCTION

The study of generalized open sets and their associated topological properties has been an active area of research in general topology. In this direction, the concept of supra topology was introduced by A. S. Mashhour et al [11] in1983. They studied s-continuous functions and s^*-continuous functions. Subsequently, R. Devi et al [6] introduced the concept of supra α-open sets and supra α-continuous mappings in 2008. Jamal M. Mustafa [15] studied supra b-compact and supra b-Lindelof spaces. Vidyarani et al [23] introduced the concept of supra N-compact, countably supra N-compact, supra N-Lindelof and investigated their relationships using the concept of continuity. In 2013, Missier and Rodrigo [13] introduced new class of sets in general topology called α-open (supra α-open) sets. In 2019, Amir A. Mohammed and Beyda S. Abdullah [14] introduced a new class of open sets in a topological space called ii-$open$ sets and investigated its fundamental properties and studied some properties and several characterizations of this class including the relation of ii-$open$ sets with many other classes of open sets. They showed that the family of ii-$open$ subsets of a topological space (Z, T) forms a topology on X which is finer than τ. In 2022, Beyda S. Abdullah, Sabih W. Askandar and Ruqayah N. Balo [3] introduced a new kind of an open set which is called an $h\alpha$-open set.

They also studied the relationship between $h\alpha$-open sets and a variety of other open sets as h-open sets, open sets, semi-open sets and α-open sets. Furthermore, they introduced the concepts of $h\alpha$-continuous mappings, $h\alpha$-open mappings, $h\alpha$-irresolute mappings, $h\alpha$-totally continuous mappings. We will extend the concept of compactness via $h\alpha$-open sets by introducing $h\alpha$-compact spaces in topological spaces and will investigate its characterizations by making use of generalized mappings including $h\alpha$-continuous functions and $h\alpha$-irresolute functions. The objective of this paper is to introduce the new concepts called $h\alpha$-compact space and $h\alpha$-Lindelöf space, countably $h\alpha$-compact space, almost $h\alpha$-compact space, and mildly $h\alpha$-compact space in topological spaces and investigate fundamental properties and characterizations of these new notions of spaces in topological spaces.

2. PRELIMINARIES

The (Z,T) and (W,Ψ) $(or\ simply,\ Z\ and\ W)$ denote topological spaces on which no separation axioms are assumed unless explicitly stated, and $\xi:(Z,\mathrm{T})\to(W,\Psi)$ means a mapping ξ from a topological space Z to a topological space W. A subset M of a topological space (Z,T) is said to be open if $M\in T$. A subset M of a topological space Z is said to be closed if the set $Z-M$ is open. The interior of a subset A of a topological space X is defined as the union of all open sets contained in A. It is denoted by $Int(A)$. The closure of a subset A of a topological space X is defined as the intersection of all closed sets containing A. It is denoted by $Cl(A)$. If U is a set and x, is a point in X, then $N(x)$, and U^c denote respectively, the neighborhood system of x, and the complement of U, respectively. We shall define the concept of $h\alpha$-open set and determine its connection to the classical open set and characterize the concepts of $h\alpha$-*open* sets.

Definition 2.1. A subset M of a topological space (Z,T) is an α-*open* set, if $M\subseteq Int\left[Cl\left(Int(M)\right)\right]$. The family of all α-*open* sets in a topological space (Z,T) is denoted by T^{α}.

Definition 2.2. A subset M of a topological space (Z,T) is said to be an h-*open* set, if for every non-empty set O in Z, $O\neq Z$. and $O\in\mathrm{T}$, such that $M\subseteq Int(M\cup O)$. The complement of h-*open* set is called h-*closed* set.

Definition 2.3. A subset M of a topological space (Z,T) is said to be an $h\alpha$-*open* set, if for every non-empty set O in Z, $O\neq Z$ and $O\in\mathrm{T}^{\alpha}$, such that $M\subseteq Int(M\cup O)$. The complement of $h\alpha$-*open* set is called $h\alpha$-*closed* set. The family of all $h\alpha$-*open* sets in a topological space (Z,T) is denoted by $\mathrm{T}^{h\alpha}$.

Proposition 2.4. (i) Every closed (open) set is $h\alpha$-closed ($h\alpha$-open) set.

(ii) Every closed (open) set is h-closed (h-open) set.

(iii) Every h-closed (h-open) set is an $h\alpha$-closed ($h\alpha$-open) set.

Definition 2.5. Let (Z,T) be a topological space and $\mathrm{M}\subseteq Z$. We define the $h\alpha$-closure of M (briefly $h\alpha$-Cl(M)) to be the intersection of all $h\alpha$-closed sets containing M which is denoted by $h\alpha\text{-}Cl(M)=\mathrm{I}\{M:M\subseteq K\ and\ K\ is\ h\alpha\text{-}closed\ in\ Z.\}$. We note that $h\alpha\text{-}Cl(M)$ is the smallest $h\alpha$-*closed* set containing M.

Definition 2.6. Let (Z,T) be any topological space and M be a subset of X. A point z of Z is called an $\mathrm{h}\alpha$-*interior* point of M, if there exists an $\mathrm{h}\alpha$-*open* set P such that $z \in P \subseteq M$. The set of all $\mathrm{h}\alpha$-*interior* points of M is said to be $\mathrm{h}\alpha$-*interior* of M $(briefly\ \mathrm{h}\alpha\text{-}Int(M))$ and it is defined by $\mathrm{h}\alpha\text{-}Int(M) = \mathrm{U}\{P : P \subseteq M \text{ and } P \in \mathrm{T}^{\mathrm{h}\alpha}\}$.

Proposition2.7. If P and Q are sets in a topological space (Z,T), then

(1) $\mathrm{h}\alpha\text{-}Int(\phi) = \phi$.

(2) $\mathrm{h}\alpha\text{-}Int(Z) = Z$.

(3) $\mathrm{h}\alpha\text{-}Int(P) \subseteq P$.

(4) $P \subseteq Q \Rightarrow \mathrm{h}\alpha\text{-}Int(P) \subseteq \mathrm{h}\alpha\text{-}Int(Q)$.

Proposition2.8. Let M be any subset of a topological space (Z,T). Then $z \in \mathrm{h}\alpha\text{-}Cl(M)$ if and only if for every $h\alpha$-*open* set P containing z, $P \mathrm{I}\ Z \neq \phi$.

Proposition2.9. For any subset P of topological space (Z,T), $\mathrm{h}\alpha\text{-}Int(P) \subseteq P \subseteq \mathrm{h}\alpha\text{-}Cl(P)$.

Definition 2.10. Let P be a subset of a topological space (Z,T). A point $p \in P$ is said to be an $\mathrm{h}\alpha$-*limit* point of P if for each $\mathrm{h}\alpha$-*open* set O containing p. $O \mathrm{I}\ (P - \{p\}) \neq \phi$. The set of all $\mathrm{h}\alpha$-*limit* points of P is called the $\mathrm{h}\alpha$-*derived* set of P and is denoted by $\mathrm{h}\alpha\text{-}D(P)$.

Theorem 2.11. For any subset P of a topological space Z, $\mathrm{h}\alpha\text{-}Cl(P) = P \mathrm{U}[\mathrm{h}\alpha\text{-}D(P)]$.

Proof. Since $\mathrm{h}\alpha\text{-}D(P) \subseteq \mathrm{h}\alpha\text{-}Cl(P)$. $P \mathrm{U}[\mathrm{h}\alpha\text{-}D(P)] \subseteq \mathrm{h}\alpha\text{-}Cl(P)$. On the other hand, let $p \in \mathrm{h}\alpha\text{-}Cl(P)$. If $p \in P$, then the proof is complete. If $p \notin P$, each $\mathrm{h}\alpha$-*open* set O containing p intersects P at a point distinct from p, so $p \in \mathrm{h}\alpha\text{-}D(P)$. Thus, $\mathrm{h}\alpha\text{-}Cl(P) \subseteq [P \mathrm{U}(\mathrm{h}\alpha\text{-}D(P))]$, which completes the proof.

Corollary 2.12. A subset P of a space Z is $\mathrm{h}\alpha$-*closed* if and only if it contains the set of all of its $\mathrm{h}\alpha$-*limit* points.

Theorem 2.13. For subsets P, Q of a space (Z,T). the following statements are true:

(1) $\mathrm{h}\alpha\text{-}Int(P)$ is the largest $\mathrm{h}\alpha$-*open* set contained in P.

(2) P is $\mathrm{h}\alpha$-*open* if and only if $P = \mathrm{h}\alpha\text{-}Int(P)$.

(3) $\mathrm{h}\alpha\text{-}Int[\mathrm{h}\alpha\text{-}Int(P)] = \mathrm{h}\alpha\text{-}Int(P)$;

(4) $\mathrm{h}\alpha\text{-}Int(P) = [P - (\mathrm{h}\alpha\text{-}D(Z - P))]$;

(5) $[Z - (\mathrm{h}\alpha\text{-}Cl(P))] = \mathrm{h}\alpha\text{-}Int(Z - P)$;

(6) $[Z - (\mathrm{h}\alpha\text{-}Int(P))] = \mathrm{h}\alpha\text{-}Cl(Z - P)$;

(7) $[h\alpha\text{-}Int(P)]\cup[h\alpha\text{-}Int(P)]\subseteq h\alpha\text{-}Int(P\cup Q)$;

(8) $h\alpha\text{-}Int(P\cap Q)=[h\alpha\text{-}Int(A)]\cap[h\alpha\text{-}Int(B)]$;

Proof. (4) If $p\in[P-(h\alpha\text{-}D(Z-P))]$, then $p\notin h\alpha\text{-}D(Z-P)$ and so there exists an $h\alpha\text{-}open$ set O containing p such that $O\cap(Z-P)=\phi$. Then, $p\in O\subseteq P$ and hence $p\in h\alpha\text{-}Int(P)$, that is, $[P-(h\alpha\text{-}D(Z-P))]\subseteq h\alpha\text{-}Int(P)$. On the other hand, if $p\in h\alpha\text{-}Int(P)$, then $p\notin h\alpha\text{-}D(Z-P)$ since $h\alpha\text{-}Int(P)$ is $h\alpha\text{-}open$ and $[(h\alpha\text{-}Int(P))\cap(Z-P)]=\phi$. Hence, $h\alpha\text{-}Int(P)=[P-(h\alpha\text{-}D(Z-P))]$.

(6) $Z-[h\alpha\text{-}Int(P)]=Z-[P-(h\alpha\text{-}D(Z-P))]=(Z-P)\cup[h\alpha\text{-}D(Z-P)]=h\alpha\text{-}Cl(Z-P)$.

Theorem 2.14. Let (Z,T) be a topological space and $P,Q\subseteq Z$. Then the following statements are true:

(1) $p\in h\alpha\text{-}Cl(P)$ if and only if for every $h\alpha\text{-}open$ subset O containing p, $O\cap P\neq\phi$.

(2) $P\subseteq Q$ implies that $h\alpha\text{-}Cl(P)\subseteq h\alpha\text{-}Cl(Q)$.

(3) P is $h\alpha\text{-}closed$ if and only if $h\alpha\text{-}Cl(P)=P$.

(4) $h\alpha\text{-}Cl[h\alpha\text{-}Cl(P)]=h\alpha\text{-}Cl(P)$.

(5) $[h\alpha\text{-}Cl(P)]\cup[h\alpha\text{-}Cl(Q)]=h\alpha\text{-}Cl(P\cup Q)$.

(6) $h\alpha\text{-}Int(Z-P)=X-[h\alpha\text{-}Cl(P)]$.

(7) $h\alpha\text{-}Cl(Z-P)=X-[h\alpha\text{-}Int(P)]$.

Definition 2.15. $h\alpha\text{-}Bd(P)=P-[h\alpha\text{-}Int(P)]$ is said to be the $h\alpha\text{-}border$ of P.

Theorem 2.16. For a subset P of a space Z, the following statements hold:

(i) $Bd(P)\subseteq h\alpha\text{-}Bd(P)$ where $Bd(P)$ denotes the border of P;

(ii) $P=h\alpha\text{-}Int(P)\cup h\alpha\text{-}Bd(P)$;

(iii) $[h\alpha\text{-}Int(P)]\cap[h\alpha\text{-}Bd(P)]=\phi$;

(iv) P is an $h\alpha\text{-}open$ set if and only if $h\alpha\text{-}Bd(P)=\phi$;

(v) $h\alpha\text{-}Bd[h\alpha\text{-}Int(P)]=\phi$;

(vi) $h\alpha\text{-}Int[h\alpha\text{-}Bd(\phi)]=\phi$;

(vii) $h\alpha\text{-}Bd[h\alpha\text{-}Bd(P)]=h\alpha\text{-}Bd(P)$;

(viii) $h\alpha\text{-}Bd(P)=P\cap[h\alpha\text{-}Cl(Z-P)]$;

(ix) $h\alpha\text{-}Bd(P)=h\alpha\text{-}D(Z-P)$.

Proof. (vi) If $p\in h\alpha\text{-}Int[h\alpha\text{-}Bd(P)]$, then $p\in h\alpha\text{-}Bd(P)$. On the other hand, since $h\alpha\text{-}Bd(P)\subseteq P$, $p\in h\alpha\text{-}Int[h\alpha\text{-}Bd(P)]\subseteq h\alpha\text{-}Int(P)$. Therefore, we get $p\in[h\alpha\text{-}Int(P)]\text{I}\ [h\alpha\text{-}Bd(P)]$, which contradicts (iii). Thus, $h\alpha\text{-}Int[h\alpha\text{-}Bd(P)]=\phi$.

(viii) $h\alpha\text{-}Bd(A)=P-[h\alpha\text{-}Int(P)]=Z-[X-(h\alpha\text{-}Cl(Z-P))]=P\text{I}\ [h\alpha\text{-}Cl(Z-P)]$.

(ix) $h\alpha\text{-}Bd(P)=P-[h\alpha\text{-}Int(P)]=P-[P-(h\alpha\text{-}D(Z-P))]=h\alpha\text{-}D(Z-P)$.

Definition 2.17. $h\alpha\text{-}Fr(P)=[h\alpha\text{-}Cl(P)]-[h\alpha\text{-}Int(P)]$ is said to be the $h\alpha\text{-}$*frontier* of P.

Theorem 2.18. For a subset P of a space (Z,T), the following statements hold:

(i) $Fr(P)\subseteq h\alpha\text{-}Fr(P)$ where $Fr(P)$ denotes the frontier of P;

(ii) $h\alpha\text{-}Cl(P)=[h\alpha\text{-}Int(P)]\text{U}[h\alpha\text{-}Fr(P)]$;

(iii) $[h\alpha\text{-}Int(P)]\text{I}\ [h\alpha\text{-}Fr(P)]=\phi$;

(iv) $h\alpha\text{-}Bd(P)\subseteq h\alpha\text{-}Fr(P)$;

(v) $h\alpha\text{-}Fr(P)=[h\alpha\text{-}Bd(P)]\text{U}[h\alpha\text{-}D(P)]$;

(vi) P is a $h\alpha\text{-}$*open* set if and only if $h\alpha\text{-}Fr(P)=h\alpha\text{-}D(P)$;

(vii) $h\alpha\text{-}Fr(P)=[h\alpha\text{-}Cl(P)]\text{I}\ [h\alpha\text{-}Cl(Z-P)]$;

(viii) $h\alpha\text{-}Fr(P)=h\alpha\text{-}Fr(Z-P)$;

(ix) $h\alpha\text{-}Fr(P)$ is $h\alpha\text{-}$*closed*;

(x) $h\alpha\text{-}Fr[h\alpha\text{-}Fr(P)]\subseteq h\alpha\text{-}Fr(P)$;

(xi) $h\alpha\text{-}Fr[h\alpha\text{-}Int(A)]\subseteq h\alpha\text{-}Fr(A)$;

(xii) $h\alpha\text{-}Fr[h\alpha\text{-}Cl(P)]\subseteq h\alpha\text{-}Fr(P)$;

(xiii) $h\alpha\text{-}Int(P)=P-[h\alpha\text{-}Fr(P)]$.

Proof. (ii) $[h\alpha\text{-}Int(P)]\text{U}[h\alpha\text{-}Fr(P)]=h\alpha\text{-}Int(P)\text{U}[h\alpha\text{-}Cl(P)-h\alpha\text{-}Int(P)]=h\alpha\text{-}Cl(P)$
(iii) $[h\alpha\text{-}Int(P)]\text{I}\ [h\alpha\text{-}Fr(P)]=[h\alpha\text{-}Int(P)]\text{I}\ [(h\alpha\text{-}Cl(P))-(h\alpha\text{-}Int(P))]=\phi$.

(v) *Since* $[h\alpha\text{-}Int(P)]\text{U}[h\alpha\text{-}Fr(P)]=[h\alpha\text{-}Int(P)]\text{U}[h\alpha\text{-}Bd(P)]\text{U}[h\alpha\text{-}D(P)]$,
$h\alpha\text{-}Fr(P)=[h\alpha\text{-}Bd(P)]\text{U}[h\alpha\text{-}D(P)]$.

(vii) $h\alpha\text{-}Fr(P)=[h\alpha\text{-}Cl(P)]-[h\alpha\text{-}Int(P)]=[h\alpha\text{-}Cl(P)]\text{I}\ [h\alpha\text{-}Cl(Z-P)]$.

(ix) $h\alpha\text{-}Cl[h\alpha\text{-}Fr(P)] = h\alpha\text{-}Cl[(h\alpha\text{-}Cl(P)) \cap (h\alpha\text{-}Cl(Z-P))] \subseteq h\alpha\text{-}Cl[h\alpha\text{-}Cl(P)] \cap h\alpha\text{-}Cl[h\alpha\text{-}Cl(Z-P)]$
$= [h\alpha\text{-}Cl(P)] \cap h\alpha\text{-}[h\alpha\text{-}Cl(Z-P)] = h\alpha\text{-}Fr(P)$. Hence $h\alpha\text{-}Fr(A)$ is $h\alpha\text{-}closed$.

(x) $h\alpha\text{-}Fr[h\alpha\text{-}Fr(A)] = h\alpha\text{-}Cl[h\alpha\text{-}Fr(A)] \cap h\alpha\text{-}Cl[X-(h\alpha\text{-}Fr(A))] \subseteq h\alpha\text{-}Cl[h\alpha\text{-}Fr(A)] = h\alpha\text{-}Fr(A)$.

(xii) $h\alpha\text{-}Fr[h\alpha\text{-}Cl(A)] = h\alpha\text{-}Cl[h\alpha\text{-}Cl(A)] - h\alpha\text{-}Int[h\alpha\text{-}Cl(A)] = [h\alpha\text{-}Cl(A)] - [h\alpha\text{-}Int(h\alpha\text{-}Cl(A))] \subseteq$
$[(h\alpha\text{-}Cl(A)) - (h\alpha\text{-}Int(A))] = h\alpha\text{-}Fr(A)$.

(xiii) $A - [h\alpha\text{-}Fr(A)] = A - [(h\alpha\text{-}Cl(A)) - (h\alpha\text{-}Int(A))] = h\alpha\text{-}Int(A)$.

Theorem 2.19. Arbitrary union of $h\alpha\text{-}open$ sets in a topological space (Z,T) is an $h\alpha\text{-}open$ set in X.

Proof. Let $\Gamma = \{A_i : i \in I\} \subseteq \tau^{h\alpha}$. Let $U \in \tau^{\alpha} - \{\phi\}$. Then for each $i \in I$, $A_i \subseteq Int(A_i \cup U)$. Hence, $\cup_{i\in} A_i \subseteq \cup_{i\in I} Int(A_i \cup U) \subseteq Int((\cup_{i\in I} A_i) \cup U)$. So, $\cup_{i\in} A_i \subseteq Int((\cup_{i\in I} A_i) \cup U)$. It follows that $\cup\Gamma = \cup\{A_i : i \in I\} = \cup_{i\in I} A_i$ is an $h\alpha\text{-}open$ set in (Z,T).

Definition 2.20. A subset A of a topological space (Z,T) is called an $h\alpha\text{-}neighbourhood$ of a point x in X if there exists an $h\alpha\text{-}open$ set U in X such that $x \in U \subseteq A$. The set of all $h\alpha\text{-}neighbourhoods$ of an element x in X will be denoted by: $h\alpha\text{-}N_x = h\alpha\text{-}N_x(X) = \{A \subseteq X : A \text{ is an } h\alpha\text{-}neighbourhood \text{ of } x\}$.

Proposition 2.21. A subset A of a topological space (Z,T) is $h\alpha\text{-}open$ if and only if it is a $h\alpha\text{-}neighbourhood$ of each of its points.

Proof. If A is $h\alpha\text{-}open$ in X, then $x \in A \subseteq A$ for each $x \in A$. Thus A is an $h\alpha\text{-}neighbourhood$ of each of its points. Conversely, suppose that A is an $h\alpha\text{-}neighbourhood$ of each of its points. Then for each $x \in A$, there exists an $h\alpha\text{-}open$ set U_x in X such that $x \in U_x \subseteq A$. Hence $\cup_{x\in A} U_x \subseteq A$. Since $A \subseteq \cup_{x\in A} U_x$, therefore $A = \cup_{x\in A} U_x$. Thus A is an $h\alpha\text{-}open$ set in X, since it is a union of $h\alpha\text{-}open$ sets

Definition 2.22. A mapping $\xi : (Z,T) \to (W,\Psi)$ is called $h\alpha\text{-}irresolute$ at a point $x \in X$ if for all $h\alpha\text{-}open$ subsets V in Y containing $f(x)$, there is an $h\alpha\text{-}open$ subset U of x such that $x \in U$ and $f(U)$ is a subset of V. The function f will be called $h\alpha\text{-}irresolute$ if f is $h\alpha\text{-}irresolute$ at each point $x \in X$.

Theorem 2.23. Let $\xi : (Z,T) \to (W,\Psi)$ $\xi : (Z,T) \to (W,\Psi)$ be a function. Then the following statements are equivalent.

(1) f is $h\alpha\text{-}irresolute$.

(2) For each $x \in X$ and each $h\alpha\text{-}neighbourhood$ V of $f(x)$ in Y, there is an $h\alpha\text{-}neighborhood$ U of x such that $f(U) \subseteq V$.

(3) The inverse image of every $h\alpha\text{-}closed$ subset of Y is an $h\alpha\text{-}closed$ subset of X.

(4) The inverse image of every $h\alpha\text{-}open$ subset of Y is an $h\alpha\text{-}open$ subset of X.

Definition 2.24. A function $\xi:(Z,\mathrm{T})\to(W,\Psi)$ is called $\mathrm{h}\alpha$-*continuous* if $f^{-1}(V)$ is $\mathrm{h}\alpha$-*open* set in X for every open set V in Y.

Proposition 2.25. A function $\xi:(Z,\mathrm{T})\to(W,\Psi)$ is called $\mathrm{h}\alpha$-*continuous* if and only if $f^{-1}(V)$ is $\mathrm{h}\alpha$-*closed* set in X for every closed set V in Y.

Definition 2.26. A function $\xi:(Z,\mathrm{T})\to(W,\Psi)$ is called *Pre*-$\mathrm{h}\alpha$--*open* if and only if the image set $f(U)$ is $\mathrm{h}\alpha$-*open* set in Y for every $\mathrm{h}\alpha$-*open* set U in X.

Proposition 2.27. A bijection function $f:(X,\tau)\to(Y,\sigma)$ is called $\mathrm{h}\alpha$-*homeomorphism* if f is *pre*-$\mathrm{h}\alpha$-*open* and $\mathrm{h}\alpha$-*irreseolute.*

Definition 2.28. A function $f:(X,\tau)\to(Y,\sigma)$ is called perfectly $\mathrm{h}\alpha$-*continuous* if the inverse image $f^{-1}(V)$ of every $\mathrm{h}\alpha$-*open* set V of Y is both open and closed in X.

Definition 2.29. A function $\xi:(Z,\mathrm{T})\to(W,\Psi)$ is called strongly $\mathrm{h}\alpha$-*continuous* if the inverse image $f^{-1}(V)$ of every $\mathrm{h}\alpha$-*open* V in Y is open in X.

3. Hα-COMPACT SPACES

Definition 3.1. A collection $\{A_i:i\in I\}$ of $\mathrm{h}\alpha$-*open* sets in a topological space (X,τ) is called an $\mathrm{h}\alpha$-*open* cover of a subset B of X if $B\subseteq \mathrm{U}\{A_i:i\in I\}$ holds.

Definition 3.2. A topological space (X,τ) is called $\mathrm{h}\alpha$-*compact* if every $\mathrm{h}\alpha$-*open* cover of X has a finite subcover.

Definition 3.3. A subset B of a topological space (X,τ) is said to be $\mathrm{h}\alpha$-*compact* relative to (X,τ) if, for every collection $\{A_i:i\in I\}$ of $\mathrm{h}\alpha$-*open* subsets of X such that $B\subseteq \mathrm{U}\{A_i:i\in I\}$ there exists a finite subset I_0 of I such that $B\subseteq \mathrm{U}\{A_i:i\in I_0\}$.

Definition 3.4. A subset E of a topological space (Z,T) is said to be $\mathrm{h}\alpha$-*compact* if E is $\mathrm{h}\alpha$-*compact* as a subspace of X.

Definition 3.5. Every $\mathrm{h}\alpha$-*compact* space is compact.

Proof. Let $\{A_i:i\in I\}$ be an open cover of (Z,T) Since every open set in X is $\mathrm{h}\alpha$-*open* in X. So $\{A_i:i\in I\}$ is an $\mathrm{h}\alpha$-*open* cover of (Z,T). Since (Z,T) is $\mathrm{h}\alpha$-*compact*, $\mathrm{h}\alpha$-*open* cover $\{A_i:i\in I\}$ of (Z,T) has a finite sub cover say $\{A_i:i=1,2,3,...,n\}$ for X. Hence (Z,T) is a compact space.

Definition 3.6. Every $\mathrm{h}\alpha$-*closed* subset of a $\mathrm{h}\alpha$-*compact* space (Z,T) is $\mathrm{h}\alpha$-*compact* relative to X.

Proof. Let A be an $\mathrm{h}\alpha$-*closed* subset of a topological space (Z,T). Then A^c is $\mathrm{h}\alpha$-*open* in (Z,T). Let $\Gamma=\{A_i:i\in I\}$ be an $\mathrm{h}\alpha$-*open* cover of A by $\mathrm{h}\alpha$-*open* subsets of (Z,T). Then $\Gamma^*=\Gamma\,\mathrm{U}\{A^c\}$ is an $\mathrm{h}\alpha$-*open* cover of (Z,T). That is $X=(\mathrm{U}_{i\in I}A_i)\mathrm{U}A^c$. By hypothesis, (X,τ) is an $\mathrm{h}\alpha$-*compact* space and hence Γ^* is reducible to a finite subcover of (X,τ) say $X=(\mathrm{U}_{i\in I_0}A_i)\mathrm{U}A^c$ for some finite subset I_0 of I. But A and A^c are

disjoint. Hence $A \subseteq U\{A_i : i \in I_0\}$. Thus $h\alpha$-*open* cover $\Gamma = \{A_i : i \in I\}$ of A contains a finite subcover. Hence A is $h\alpha$-*compact* relative to (Z,T).

Theorem 3.7. An $h\alpha$-*continuous* image of an $h\alpha$-*compact* space is compact.

Proof. Let $\xi : (Z,T) \to (W,\Psi)$ be an $h\alpha$-*continuous* mapping from an $h\alpha$-*compact* topological space (Z,T) onto a topological space (Y,σ). Let $\Gamma = \{A_i : i \in I\}$ be an open cover of Y. Therefore $f^{-1}(\Gamma) = \{f^{-1}(A_i) : i \in I\}$ is an $h\alpha$-*open* cover of X, as f is $h\alpha$-*continuous*. Since X is $h\alpha$-*compact*, the $h\alpha$-*open* cover $f^{-1}(\Gamma) = \{f^{-1}(A_i) : i \in I\}$ of X, has a finite subcover say $\{f^{-1}(A_i) : i = 1,2,3,...,n\}$. Therefore $X = U_{i=1}^{n} f^{-1}(A_i)$, which implies $Y = f(X) = U_{i=1}^{n} A_i$. That is $\{A_i : i = 1,2,3,...,n\}$ is a finite subcover of $\Gamma = \{A_i : i \in I\}$. Hence (W,Ψ) is compact.

Theorem 3.8. Suppose that a function $\xi : (Z,T) \to (W,\Psi)$ is $h\alpha$-*irresolute* and a subset S of X is $h\alpha$-*compact* relative to (Z,T), then the image $f(S)$ is $h\alpha$-*compact* relative to (Y,σ).

Proof. Let $\Gamma = \{A_i : i \in I\}$ be a collection of $h\alpha$-*open* subsets of (Y,σ), such that $f(S) \subseteq U\{A_i : i \in I\}$. Since f is $h\alpha$-*irresolute*. So $S \subseteq U\{f^{-1}(A_i) : i \in I\}$, where $\{f^{-1}(A_i) : i \in I\} \subseteq \tau^{h\alpha}$. Since S is $h\alpha$-*compact* relative to (X,τ), there exists a finite sub collection $\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\}$ such that $S \subseteq U\{f^{-1}(A_1), f^{-1}(A_2), \ldots, f^{-1}(A_n)\}$. That is $f(S) \subseteq U\{A_1, A_2, ..., A_n\}$. Hence $f(S)$ is $h\alpha$-*compact* relative to (Y,σ).

Theorem 3.9. Suppose that a mapping $\xi : (Z,T) \to (W,\Psi)$ is strongly $h\alpha$-*continuous* mapping from a compact space (X,τ) onto a topological space (Y,σ). Then (Y,σ) is $h\alpha$-*compact*.

Proof. Let $\{A_i : i \in I\}$ be an $h\alpha$-*open* cover of (Y,σ). Since f is strongly $h\alpha$-*continuous*, $\{f^{-1}(A_i) : i \in I\}$ is an open cover of (Z,T). Again, since (Z,T) is compact, the open cover $\{f^{-1}(A_i) : i \in I\}$ of (X,τ) has a finite sub cover say $\{f^{-1}(A_i) : i = 1,2,3,...,n\}$. Therefore $X = U\{f^{-1}(A_i) : i = 1,2,3,...,n\}$, which implies $f(X) = U\{A_i : i = 1,2,3,...,n\}$, so that $Y = U\{A_i : i = i = 1,2,3,...,n\}$. Thus $\{A_1, A_2, ..., A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for (Y,σ). Hence (Y,σ) is $h\alpha$-*compact*.

Theorem 3.10. Suppose that a mapping $\xi : (Z,T) \to (W,\Psi)$ is perfectly $h\alpha$-*continuous* mapping from a compact space (Z,T) onto a topological space (W,Ψ) Then (W,Ψ) is $h\alpha$-*compact*.

Proof. Let $\{A_i : i \in I\}$ be an $h\alpha$-*open* cover of (W,Ψ) Since f is perfectly $h\alpha$-*continuous*, $\{f^{-1}(A_i) : i \in I\}$ is an open cover of (Z,T). Again, since (X,τ) is compact, the open cover $\{f^{-1}(A_i) : i \in I\}$ of (X,τ) has a finite sub cover say $\{f^{-1}(A_i) : i = 1,2,3,...,n\}$. Therefore $X = U\{f^{-1}(A_i) : i = 1,2,3,...,n\}$, which implies $f(X) = U\{A_i : i = 1,2,3,...,n\}$, so that $Y = U\{A_i : i = 1,2,3,...,n\}$. That is $\{A_1, A_2, \ldots, A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for (Y,σ). Hence (Y,σ) is $h\alpha$-*compact*.

Theorem 3.11. Suppose that a function $\xi : (Z,T) \to (W,\Psi)$ is an $h\alpha$-*irresolute* mapping from an $h\alpha$-*compact* space (Z,T) onto a topological space (Y,σ). Then (Y,σ) is $h\alpha$-*compact*.

Proof. Let $\xi:(Z,\mathrm{T})\to(W,\Psi)$ be an $\mathrm{h}\alpha$-*irresolute* mapping from an $\mathrm{h}\alpha$-*compact* space (X,τ) onto a topological space (Y,σ). Let $\{A_i : i\in I\}$ be an $\mathrm{h}\alpha$-*open* cover of (W,Ψ). Then $\{f^{-1}(A_i) : i\in I\}$ is an $\mathrm{h}\alpha$-*open* cover of (Z,T) since f is $\mathrm{h}\alpha$-*irresolute*. As (X,τ) is $\mathrm{h}\alpha$-*compact*, the $\mathrm{h}\alpha$-*open* cover $\{f^{-1}(A_i) : i\in I\}$ of (Z,T) has a finite sub cover say $\{f^{-1}(A_i) : i=1,2,3,...,n\}$. Therefore $X=\mathrm{U}\{f^{-1}(A_i) : i=1,2,3,...,n\}$, which implies $f(X)=\mathrm{U}\{A_i : i=1,2,3,...,n\}$, so that $Y=\mathrm{U}\{A_i : i=1,2,3,...,n\}$. That is $\{A_1, A_2, \ldots, A_n\}$ is a finite subcover of $\{A_i : i\in I\}$ for (Y,σ). Hence (Y,σ) is $\mathrm{h}\alpha$-*compact*.

Theorem 3.12. If (Z,T) is compact and every $\mathrm{h}\alpha$-*closed* set in X is also closed in X, then (X,τ) is $\mathrm{h}\alpha$-*compact*.

Proof. Let $\{A_i : i\in I\}$ be an $\mathrm{h}\alpha$-*open* cover of X. Since every $\mathrm{h}\alpha$-*closed* set in X is also closed in X. Thus $\{X-A_i : i\in I\}$ is a closed cover of X and hence $\{A_i : i\in I\}$ is an open cover of X. Since (X,τ) is compact. So there exists a finite subcover $\{A_i : i=1,2,3,...,n\}$ of $\{\mathrm{A_i} : \mathrm{i}\in \mathrm{I}\}$ such that $X=\mathrm{U}\{A_i : i=1,2,3,...,n\}$. Hence (X,τ) is $\mathrm{h}\alpha$-*compact*.

Theorem 3.13. A topological space (Z,T) is $\mathrm{h}\alpha$-*compact* if and only if every family of $\mathrm{h}\alpha$-*closed* sets of (X,τ) having finite intersection property has a nonempty intersection.

Proof. Suppose (Z,T) is $\mathrm{h}\alpha$-*compact*. Let $\{A_i : i\in I\}$ be a family of $\mathrm{h}\alpha$-*closed* sets with finite intersection property. Suppose $\mathrm{I}_{i\in I} A_i=\phi$, then $X-\mathrm{I}(\{A_i : i\in I\})=X$. This implies $\mathrm{U}\{(X-A_i) : i\in I\}=X$. Thus the cover $\{(X-A_i) : i\in I\}$ is an $\mathrm{h}\alpha$-*open* cover of (X,τ). Then as (X,τ) is $\mathrm{h}\alpha$-*compact*, the $\mathrm{h}\alpha$-*open* cover $\{(X-A_i) : i\in I\}$ has a finite sub cover say $\{(X-A_i) : i=1,2,3,...,n\}$. This implies that $X=\mathrm{U}\{(X-A_i) : i=1,2,3,...,n\}$ which implies $X=X-\mathrm{I}\{A_i : i=1,2,3,...,n\}$, which implies $X-X=X-\left[X-\mathrm{I}\{A_i : i=1,2,3,...,n\}\right]$, which implies $\phi=\mathrm{I}\{A_i : i=1,2,3,...,n\}$. This disproves the assumption. Hence $\mathrm{I}\{A_i : i\in I\}\neq\phi$.

Conversely, suppose (Z,T) is not $\mathrm{h}\alpha$-*compact*. Then there exits an $\mathrm{h}\alpha$-*open* cover of (Z,T) say $\{G_i : i\in I\}$ having no finite subcover. This implies for any finite subfamily $\{G_i : i=1,2,3,...,n\}$ of $\{G_i : i\in I\}$, we have $\mathrm{U}\{G_i : i=1,2,3,...,n\}\neq X$, which implies $X-(\mathrm{U}\{G_i : i=1,2,3,...,n\})\neq X-X$, therefore $\mathrm{I}\{X-G_i : i=1,2,3,...,n\}\neq\phi$. Then the family $\{X-G_i : i\in I\}$ of $\mathrm{h}\alpha$-*closed* sets has a finite intersection property. Also by assumption $\mathrm{I}\{X-G_i : i\in I\}\neq\phi$ which implies $X-(\mathrm{U}\{G_i : i\in I\})\neq\phi$, so that $\mathrm{U}\{G_i : i\in I\}\neq X$. This implies $\{G_i : i\in I\}$ is not a cover of (X,τ). This disproves the fact that $\{G_i : i\in I\}$ is a cover for (X,τ). Therefore an $\mathrm{h}\alpha$-*open* cover $\{G_i : i\in I\}$ of (X,τ) has a finite sub cover $\{G_i : i=1,2,3,...,n\}$. Hence (X,τ) is $\mathrm{h}\alpha$-*compact*.

Theorem 3.14. Let A be an $\mathrm{h}\alpha$-*compact* set relative to a topological space X and B be an $\mathrm{h}\alpha$-*closed* subset of X. Then $A\mathrm{I}\ B$ is $\mathrm{h}\alpha$-*compact* relative to X.

Proof. Let A be $\mathrm{h}\alpha$-*compact* relative to X. Let $\{A_i : i\in I\}$ be a cover of $A\mathrm{I}\ B$ by $\mathrm{h}\alpha$-*open* sets in X. Then $\{A_i : i\in I\}\mathrm{U}\{B^C\}$ is a cover of A by $\mathrm{h}\alpha$-*open* sets in X, but A is $\mathrm{h}\alpha$-*compact* relative to X, so there exists a finite subset $I_0=\{i_1, i_2, i_3, \ldots, i_n\}\subseteq I$ such that $A\subseteq\left(\mathrm{U}\left\{A_{i_k} : k=1,2,3,\ldots,n\right\}\right)\mathrm{U}\,B^C$. Then $A\mathrm{I}\ B\subseteq\mathrm{U}\left\{A_{i_k}\ \mathrm{I}\ B : k=1,2,3,\ldots,n\right\}\subseteq\mathrm{U}\left\{A_{i_k} : k=1,2,3,...,n\right\}$. Hence $A\mathrm{I}\ B$ is $\mathrm{h}\alpha$-*compact*.

Theorem 3.15. Suppose that a function $\xi:(Z,\mathrm{T})\to(W,\Psi)$ is $\mathrm{h}\alpha$-*irresolute* and a subset B of X is $\mathrm{h}\alpha$-*compact* relative to X. Then $f(B)$ is $\mathrm{h}\alpha$-*compact* relative to Y.

Proof. Let $\{A_i : i\in I\}$ be a cover of $f(B)$ by $\mathrm{h}\alpha$-*open* subsets of Y. Since f is $\mathrm{h}\alpha$-*irresolute*. Then $\{f^{-1}(A_i) : i\in I\}$ is a cover of B by $\mathrm{h}\alpha$-*open* subsets of X. Since B is $\mathrm{h}\alpha$-*compact* relative to X, $\{f^{-1}(A_i) : i\in I\}$ has a finite subcover, say $\{f^{-1}(A_1), f^{-1}(A_2),\ldots, f^{-1}(A_n)\}$ for B. Then it implies that $\{A_i : i=1,2,3,\ldots,n\}$ is a finite subcover of $\{A_i : i\in I\}$ for $f(B)$. So $f(B)$ is $\mathrm{h}\alpha$-*compact* relative to Y.

Definition 3.16. Let (Z,T) be a topological space, and let E be a subset of X. Let $\tau_E^{\mathrm{h}\alpha}=\{A\cap E : A\in\tau^{\mathrm{h}\alpha}\}$. Then $\left(E,\tau_E^{\mathrm{h}\alpha}\right)$ is a topological space.

Theorem 3.17. Let (X,τ) be a topological space and let E be a subset of X. Then $\left(E,\tau_E^{\mathrm{h}\alpha}\right)$ is compact if and only if for any $\mathrm{h}\alpha$-*open* cover Γ of E has a finite subcover of E.

Proof. Suppose E is compact. Let $\Gamma\subseteq\tau^{\mathrm{h}\alpha}$ such that $E\subseteq\cup\Gamma$. Let $\Gamma_E=\{A\cap E : A\in\Gamma\}$. Then $E=\cup\Gamma_E$ and $\Gamma_E\subseteq\tau_E^{\mathrm{h}\alpha}$. By hypothesis, there exists a finite subset $\Gamma_E^{*}=\{A_i\cap E : i=1,2,3,\ldots,n\}\subseteq\Gamma_E$ such that $E=\cup\Gamma_E^{*}$. Then $\Gamma^{*}=\{A_i : i=1,2,3,\ldots,n\}\subseteq\Gamma$ and $E\subseteq\cup\Gamma^{*}$.

Conversely, let $\Upsilon=\{A_i\cap E : i\in I\}\subseteq\tau_E^{\mathrm{h}\alpha}$ such that $E=\cup\Upsilon$. Then $\Upsilon^{*}=\{A_i : i\in A\}$ is an $\mathrm{h}\alpha$-*open* covering of E. By hypothesis, there exists $\Upsilon^{**}=\{A_i : i=1,2,3,\ldots,n\}$ a finite subset of Υ^{*} such that $E\subseteq\cup\Upsilon^{**}$. Then $\Upsilon^{\#}=\{A_i\cap E : i=1,2,3,\ldots,n\}$ is a finite subset of Υ such that $E=\cup\Upsilon^{\#}$. This proves that $\left(E,\tau_E^{\mathrm{h}\alpha}\right)$ is compact.

4. COUNTABLY H_α-COMPACT SPACES

In this section, we present the concept of countably $\mathrm{h}\alpha$-*compactness* and its properties.

Definition 4.1. A topological space (Z,T) is said to be countably $\mathrm{h}\alpha$-*compact* if every countable $\mathrm{h}\alpha$-*open* cover of X has a finite subcover.

Theorem 4.2. If (Z,T) is a countably $\mathrm{h}\alpha$-*compact* space, then (Z,T) is countably compact.

Proof. Let (Z,T) be a countably $\mathrm{h}\alpha$-*compact* space. Let $\{A_i : i\in I\}$ be a countable open cover of (Z,T). Since $\tau\subseteq\tau^{\mathrm{h}\alpha}$. So $\{A_i : i\in I\}$ is a countable $\mathrm{h}\alpha$-*open* cover of (Z,T). Since (Z,T) is countably $\mathrm{h}\alpha$-*compact*, therefore countable $\mathrm{h}\alpha$-*open* cover $\{A_i : i\in I\}$ of (X,τ) has a finite subcover, say $\{A_i : i=1,2,3,\ldots,n\}$ for X. Hence (Z,T) is a countably compact space.

Theorem 4.3. If (Z,T) is countably compact and every $\mathrm{h}\alpha$-*closed* subset of X is closed in X, then (X,τ) is countably $\mathrm{h}\alpha$-*compact*.

Proof. Let (Z,T) be a countably compact space. Let $\{A_i : i\in I\}$ be a countable $\mathrm{h}\alpha$-*open* cover of (X,τ). Since every $\mathrm{h}\alpha$-*closed* subset of X is closed in X. Thus every $\mathrm{h}\alpha$-*open* set in X is open in X. Therefore $\{A_i : i\in I\}$

is a countable open cover of (X,τ). Since (X,τ) is countably compact, so countable open cover $\{A_i : i \in I\}$ of (X,τ) has a finite subcover, say $\{A_i : i = 1,2,3,...,n\}$ for X. Hence (X,τ) is a countably $h\alpha$-*compact* space.

Theorem 4.4. Every $h\alpha$-*compact* space is countably $h\alpha$-*compact*.

Proof. Let (Z,T) be an $h\alpha$-*compact* space. Let $\{A_i : i \in I\}$ be a countable $h\alpha$-*open* cover of (Z,T). Since (Z,T) is $h\alpha$-*compact*, so $h\alpha$-*open* cover $\{A_i : i \in I\}$ of (Z,T) has a finite subcover, say $\{A_i : i = 1,2,3,...,n\}$ for (Z,T). Hence (Z,T) is countably $h\alpha$-*compact* space.

Theorem 4.5. Let $\xi : (Z,T) \to (W,\Psi)$ be an $h\alpha$-*continuous* onjective mapping. If X is countably $h\alpha$-*compact* space, then (Y,σ) is countably compact.

Proof. Let $\xi : (Z,T) \to (W,\Psi)$ be an $h\alpha$-*continuous* mapping from a countably $h\alpha$-*compact* space (X,τ) onto a topological space (Y,σ). Let $\{A_i : i \in I\}$ be a countable open cover of Y. Then $\{f^{-1}(A_i) : i \in I\}$ is a countable $h\alpha$-*open* cover of X, as f is $h\alpha$-*continuous*. Since X is countably $h\alpha$-*compact*, the countable $h\alpha$-*open* cover $\{f^{-1}(A_i) : i \in I\}$ of X has a finite subcover, say $\{f^{-1}(A_i) : i = 1,2,3,...,n\}$. Therefore $X = U\{f^{-1}(A_i) : i = 1,2,3,...,n\}$, which implies $Y = f(X) = U\{A_i : i = 1,2,3,...,n\}$. That is $\{A_i : i = 1,2,3,...,n\}$ is a finite subcover of $\{A_i : i \in I\}$ for Y. Hence Y is countably compact.

Theorem 4.6. Suppose that a mapping $\xi : (Z,T) \to (W,\Psi)$ is perfectly $h\alpha$-*continuous* mapping from a countably compact space (Z,T) onto a topological space (Y,σ). Then (Y,σ) is countably $h\alpha$-*compact*.

Proof. Let $\{A_i : i \in I\}$ be a countable $h\alpha$-*open* cover of (Y,σ). Since f is perfectly $h\alpha$-*continuous*, $\{f^{-1}(A_i) : i \in I\}$ is a countable open cover of (Y,σ). Again, since (Z,T) is countably $h\alpha$-*compact*, the countable open cover $\{f^{-1}(A_i) : i \in I\}$ of (X,τ) has a finite subcover, say $\{f^{-1}(A_i) : i = 1,2,3,...,n\}$. Therefore $X = U\{f^{-1}(A_i) : i = 1,2,3,...,n\}$, which implies $f(X) = U\{A_i : i = 1,2,3,...,n\}$, so that $Y = U\{A_i : i = 1,2,3,...,n\}$. That is $\{A_1, A_2, ..., A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for (Y,σ). Hence (Y,σ) is countably $h\alpha$-*compact*.

Theorem 4.7. Suppose that a mapping $\xi : (Z,T) \to (W,\Psi)$ is strongly $h\alpha$-*continuous* mapping from a countably compact space (Z,T) onto a topological space (W,Ψ). Then (W,Ψ) is countably $h\alpha$-*compact*.

Proof. Let $\{A_i : i \in I\}$ be a countable $h\alpha$-*open* cover of (Y,σ). Since f is strongly $h\alpha$-*continuous*, $\{f^{-1}(A_i) : i \in I\}$ is a countable open cover of (Z,T). Again, since (Z,T) is countably compact, the countable open cover $\{f^{-1}(A_i) : i \in I\}$ of (Z,T) has a finite sub cover say $\{f^{-1}(A_i) : i = 1,2,3,...,n\}$. Therefore $X = U\{f^{-1}(A_i) : i = 1,2,3,...,n\}$, which implies $f(X) = U\{A_i : i = 1,2,3,...,n\}$, so that $Y = U\{A_i : i = 1,2,3,...,n\}$. That is $\{A_1, A_2, ..., A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for (Y,σ). Hence (Y,σ) is countably $h\alpha$-*compact*.

Theorem 4.8. The image of a countably $h\alpha$-*compact* space under an $h\alpha$-*irresolute* mapping is countably $h\alpha$-*compact*.

Proof. Suppose that a mapping $\xi : (Z,T) \to (W,\Psi)$ is an $h\alpha$-*irresolute* mapping from a countably $h\alpha$-*compact* space (Z,T) onto a topological space (W,Ψ) Let $\{A_i : i \in I\}$ be a countable $h\alpha$-*open* cover of (Y,σ). Then $\{f^{-1}(A_i) : i \in I\}$ is a countable $h\alpha$-*open* cover of (X,τ), since f is $h\alpha$-*irresolute*. As (X,τ) is countably

$h\alpha$-*compact*, the countable $h\alpha$-*open* cover $\{f^{-1}(A_i) : i \in I\}$ of (X, τ) has a finite sub cover say $\{f^{-1}(A_i) : i = 1,2,3,...,n\}$. Then it follows that $X = U\{f^{-1}(A_i) : i = 1,2,3,...,n\}$, which implies $f(X) = U\{A_i : i = 1,2,3,...,n\}$, so that $Y = U\{A_i : i = 1,2,3,...,n\}$. That is $\{A_1, A_2, \ldots, A_n\}$ is a finite subcover of $\{A_i : i \in I\}$ for (Y, σ). Hence (Y, σ) is countably $h\alpha$-*compact*.

Theorem 4.9. Let (Z,T) be a topological space and $x \in X$. A point $x \in X$ is said to be $h\alpha$-*limit* point of $A \subseteq X$ provided that every $h\alpha$-*neighbourhood* of x contains at least one point of A different from x.

Theorem 4.10. Every infinite subset of an $h\alpha$-*compact* space has an $h\alpha$-*limit* point.

Proof. Let A be an infinite subset of an $h\alpha$-*compact* space (Z,T). Suppose that A has not an $h\alpha$-*limit* point. Then for each $x \in X$, there exists an $h\alpha$-*open* set G_x containing at most one point of A. Now, the collection $\Lambda = \{G_x : x \in X\}$ forms an $h\alpha$-*open* cover of X. As X is $h\alpha$-*compact*, then there exist $x_1, x_2, x_3, ..., x_n$ in X such that $X = U_{i=1}^{i=n} G_{x_i}$. Therefore X has at most n points of A. This implies that A is finite. But this contradicts that A is infinite. Thus A has an $h\alpha$-*limit* point.

5. H_α-LINDELOF SPACES

In this section, we concentrate on the concept of $h\alpha$-*Lindelöf* space and its properties.

Definition 5.1. A topological space (Z,T) is said to be an $h\alpha$-*Lindelöf* space if every $h\alpha$-*open* cover of X has a countable subcover.

Definition 5.2. Every $h\alpha$-*Lindelöf* space (Z,T) is *Lindelöf* space.

Proof. Let (Z,T) be an $h\alpha$-*Lindelöf* space. Let $\{A_i : i \in I\}$ be an open cover of (X, τ). Since $\tau \subseteq \tau^{h\alpha}$. Therefore $\{A_i : i \in I\}$ is an $h\alpha$-*open* cover of (Z,T). Since (Z,T) is $h\alpha$-*Lindelöf* space. So there exists a countable subset I_0 of I such that $\{A_i : i \in I_0\}$ is an $h\alpha$-*open* subcover of (Z,T). Hence (Z,T) is a *Lindelöf* space.

Theorem 5.3. Every $h\alpha$-*compact* space is $h\alpha$-*Lindelöf*.

Proof. Let (Z,T) be an $h\alpha$-*compact* space. Let $\{A_i : i \in I\}$ be an $h\alpha$-*open* cover of (Z,T) Since (Z,T) is $h\alpha$-*compact* space. Then $\{A_i : i \in I\}$ has a finite sub cover say $\{A_i : i = 1,2,3,...,n\}$. Since every finite subcover is always countable subcover and therefore $\{A_i : i = 1,2,3,...,n\}$ is a countable subcover of $\{A_i : i \in I\}$. Hence (X, τ) is $h\alpha$-*Lindelöf* space.

Theorem 5.4. Every $h\alpha$-*closed* subset of an $h\alpha$-*Lindelöf* space is $h\alpha$-*Lindelöf*.

Proof. Let F be an $h\alpha$-*closed* subset of X and $\{G_i : i \in I\}$ be $h\alpha$-*open* cover of F. Then F^c is $h\alpha$-*open* and $F \subseteq U\{G_i : i \in I\}$. Hence $X = (U\{G_i : i \in I\}) U F^c$. Since X is $h\alpha$-*Lindelöf*, then $X = (U\{G_i : i \in I_0\}) U F^c$ for some countable subset I_0 of I. Therefore $F \subseteq U\{G_i : i \in I_0\}$. Thus F is $h\alpha$-*Lindelöf*.

Theorem 5.5. Let A be an $h\alpha$-*Lindelöf* subset of X and B be an $h\alpha$-*closed* subset of X. Then $A \cap B$ is $h\alpha$-*Lindelöf*.

Proof. Let $\{G_i : i \in I\}$ be an $h\alpha$-*open* cover of $A \cap B$. Then $A \subseteq (\cup_{i\in I} G_i) \cup B^c$. Since A is $h\alpha$-*Lindelöf*, then there exists a countable subset I_0 of I such that $A \subseteq \left(\cup_{i\in I_0} G_i\right) \cup B^c$. Therefore $A \cap B \subseteq \cup_{i\in I_0} G_i$. Thus $A \cap B$ is $h\alpha$-*Lindelöf*.

Theorem 5.6. A topological space (Z, T) is $h\alpha$-*Lindelöf* if and only if every collection of $h\alpha$-*closed* subsets of X satisfying the countable intersection property, has, itself, a non-empty intersection.

Necessity: Let $\Lambda = \{F_i : i \in I\}$ be a collection of $h\alpha$-*closed* subsets of X which has the countable intersection property. Assume that $\cap_{i\in I} F_i = \phi$. Then $X = \cup_{i\in I} F_i^c$. Since X is $h\alpha$-*Lindelöf*, then there exists a countable subset l_0 of I such that $X = \cup_{i\in I_0} F_i^c$. Therefore, $\cap_{i\in I_0} F_i = \phi$ contradicts that Λ has the countable intersection property. Thus Λ has, itself, a non-empty intersection.

Sufficiency: Let $\{G_i : i \in I\}$ be an $h\alpha$-*open* cover of X. Suppose $\{G_i : i \in I\}$ has no countable subcover. Then $X - \cup_{i\in J} G_i \neq \phi$, for any countable subset J of I. Now, $\cap_{i\in J} G_i^c \neq \phi$ implies that $\{G_i^c : i \in I\}$ is a collection of $h\alpha$-*closed* subsets of X which has the countable intersection property. Therefore $\cap_{i\in I} G_i^c \neq \phi$. Thus $X \neq \cup_{i\in I} G_i$ contradicts that $\{G_i : i \in I\}$ is an $h\alpha$-*open* cover of X. Hence X is $h\alpha$-*Lindelöf*.

Theorem 5.7. An $h\alpha$-*continuous* image of an $h\alpha$-*Lindelöf* space is a *Lindelöf* space.

Proof. Let $\xi : (Z, T) \to (W, \Psi)$ be an $h\alpha$-*continuous* mapping from an $h\alpha$-*Lindelöf* space X onto a topological space Y. Let $\{A_i : i \in I\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i \in I\}$ is an $h\alpha$-*open* cover of X, as f is $h\alpha$-*continuous*. Since X is $h\alpha$-*Lindelöf* space, the $h\alpha$-*open* cover $\{f^{-1}(A_i) : i \in I\}$ of X has a countable subcover say $\{f^{-1}(A_i) : i \in I_0\}$ for some countable set $I_0 \subseteq I$. Therefore $X = \cup\{f^{-1}(A_i) : i \in I_0\}$, which implies $f(X) = \cup\{A_i : i \in I_0\}$, then $Y = \cup\{A_i : i \in I_0\}$. That is $\{A_i : i \in I_0\}$ a countable subcover of $\{A_i : i \in I\}$ for Y. Hence (W, Ψ) is a *Lindelöf* space.

Theorem 5.8. The image of an $h\alpha$-*Lindelöf* space under an $h\alpha$-*irresolue* mapping is $h\alpha$-*Lindelöf*. space.

Proof. Suppose that a mapping $\xi : (Z, T) \to (W, \Psi)$ is an $h\alpha$-*irresolue* mapping from an $h\alpha$-*Lindelöf* space (Z, T) onto a topological space (Y, σ). Let $\{B_i : i \in I\}$ be an $h\alpha$-*open* cover of (Y, σ). Since f is $h\alpha$-*irresolue*. Therefore $\{f^{-1}(B_i) : i \in I\}$ is an $h\alpha$-*open* cover of (X, τ). As (X, τ) is $h\alpha$-*Lindelöf* space, the $h\alpha$-*open* cover $\{f^{-1}(B_i) : i \in I\}$ of (Z, T) has a countable subcover, say $\{f^{-1}(B_i) : i \in I_0\}$ for some countable set $I_0 \subseteq I$. Therefore $X = \cup\{f^{-1}(B_i) : i \in I_0\}$, which implies $f(X) = \cup\{B_i : i \in I_0\}$, so that $Y = \cup\{B_i : i \in I_0\}$. That is $\{B_i : i \in I_0\}$ a countable subcover of $\{B_i : i \in I\}$ for Y. Hence (W, Ψ) is an $h\alpha$-*Lindelöf* space.

Theorem 5.9. If (Z, T) is $h\alpha$-*Lindelöf* space and countably $h\alpha$-*compact* space, then (Z, T) is $h\alpha$-*compact* space.

Proof. Suppose (Z, T) is $h\alpha$-*Lindelöf* space and countably $h\alpha$-*compact* space. Let $\{A_i : i \in I\}$ be an $h\alpha$-*open* cover of (X, τ). Since (Z, T) is $h\alpha$-*Lindelöf* space, $\{A_i : i \in I\}$ has a countable subcover, say $\{A_i : i \in I_0\}$ for some countable set $I_0 \subseteq I$. Therefore $\{A_i : i \in I_0\}$ is a countable $h\alpha$-*open* cover of (X, τ). Again, since (Z, T) is countably $h\alpha$-*compact* space, $\{A_i : i \in I_0\}$ has a finite subcover and say $\{A_i : i = 1, 2, 3, \ldots, n\}$. Therefore $\{A_i : i = 1, 2, 3, \ldots, n\}$ is a finite subcover of $\{A_i : i \in I\}$ for (X, τ). Hence (Z, T) is an $h\alpha$-*compact* space.

Theorem 5.10. If a function $\xi:(Z,\mathrm{T})\to(W,\Psi)$ is $\mathrm{h}\alpha$-*irresolue* and a subset A of X is $\mathrm{h}\alpha$-*Lindelöf* relative to X, then $f(A)$ is $\mathrm{h}\alpha$-*Lindelöf* relative to Y.

Proof. Let $\{B_i : i\in I\}$ be a cover of $f(A)$ by $\mathrm{h}\alpha$-*open* subsets of Y. By hypothesis, f is $\mathrm{h}\alpha$-*irresolue* and so $\{f^{-1}(B_i) : i\in I\}$ is a cover of A by $\mathrm{h}\alpha$-*open* subsets of X. Since A is $\mathrm{h}\alpha$-*Lindelöf* relative to X, $\{f^{-1}(B_i) : i\in I\}$ has a countable subcover say $\{f^{-1}(B_i) : i\in I_0\}$ for A, where I_0 is a countable subset of I. Now $\{B_i : i\in I_0\}$ is a countable subcover of $\{B_i : i\in I\}$ for $f(A)$. So $f(A)$ is $\mathrm{h}\alpha$-*Lindelöf* relative to Y.

6. ALMOST H_α-COMPACT SPACES

Definition 6.1. A topological space (Z,T) is called almost $\mathrm{h}\alpha$-*compact* (*resp. almost* $\mathrm{h}\alpha$-*Lindelöf*) provided that every $\mathrm{h}\alpha$-*open* cover of X has a finite (resp. countable) sub-collection, the $\mathrm{h}\alpha$-*closure* of whose members cover X.

The proofs of the following four propositions are straightforward and therefore will be omitted.

Proposition 6.2. Every almost $\mathrm{h}\alpha$-*compact* space is almost $\mathrm{h}\alpha$-*Lindelöf* space.

Proposition 6.3. Every $\mathrm{h}\alpha$-*compact* space (*resp.* $\mathrm{h}\alpha$-*Lindelöf space*) is almost $\mathrm{h}\alpha$-*compact* (*resp. almost* $\mathrm{h}\alpha$-*Lindelöf*).

Proposition 6.4. Any finite (resp. countable) topological space (Z,T) is almost $\mathrm{h}\alpha$-*compact* (*resp. almost* $\mathrm{h}\alpha$-*Lindelöf*).

Proposition 6.5. A finite (resp. countable) union of almost $\mathrm{h}\alpha$-*compact* (*resp. almost* $\mathrm{h}\alpha$-*Lindelöf*) subsets of (X,τ) is almost $\mathrm{h}\alpha$-*compact* (*resp. almost* $\mathrm{h}\alpha$-*Lindelöf*).

Definition 6.6. A subset E of (Z,T) is called $\mathrm{h}\alpha$-*clopen* provided that it is $\mathrm{h}\alpha$-*open* and $\mathrm{h}\alpha$-*closed*.

Theorem 6.7. Let F be an $\mathrm{h}\alpha$-*clopen* subset of an almost $\mathrm{h}\alpha$-*compact* (*resp. almost* $\mathrm{h}\alpha$-*Lindelöf*) space (X,τ). Then F is almost $\mathrm{h}\alpha$-*compact* (*resp. almost* $\mathrm{h}\alpha$-*Lindelöf*).

Proof. Let F be an $\mathrm{h}\alpha$-*clopen* subset of an almost $\mathrm{h}\alpha$-*compact* space X and $\{G_i : i\in I\}$ be an $\mathrm{h}\alpha$-*open* cover of F. Then F^c is $\mathrm{h}\alpha$-*open* and $X\subseteq\left(\mathrm{U}\{G_i : i\in I\}\right)\mathrm{U}\,F^c$. Since X is almost $\mathrm{h}\alpha$-*compact*, then there exists a finite subset I_0 of I such that $X=\left(\mathrm{U}\{\mathrm{h}\alpha\text{-}Cl(G_i) : i\in I_0\}\right)\mathrm{U}\,F^c$. Thus it follows that $F\subseteq\mathrm{U}\{\mathrm{h}\alpha\text{-}Cl(G_i) : i\in I_0\}$. Hence F is almost $\mathrm{h}\alpha$-*compact*.

The proof is similar in the case of almost $\mathrm{h}\alpha$-*Lindelöf*.

Theorem 6.8. If A is an almost $\mathrm{h}\alpha$-*compact* (*resp. almost* $\mathrm{h}\alpha$-*Lindelöf*) subset of (Z,T) and B is an $\mathrm{h}\alpha$-*clopen* subset of X, then $A\mathrm{I}\ B$ is almost $\mathrm{h}\alpha$-*compact* (*resp. almost* $\mathrm{h}\alpha$-*Lindelöf*).

Proof. Let $64\Lambda=\{G_i : i\in I\}$ be an $h\alpha$-*open* cover of $A\cap B$. Then $A\subseteq\left(\cup\{G_i : i\in I\}\right)\cup B^c$. Since A is almost $h\alpha$-*compact*, then there exists a finite subset I_0 of I such that $A\subseteq\left(\cup\{h\alpha\text{-}Cl(G_i) : i\in I_0\}\right)\cup B^c$. Hence $A\cap B\subseteq\cup\{h\alpha\text{-}Cl(G_i) : i\in I_0\}$. Thus $A\cap B$ is almost $h\alpha$-*compact*.

The proof is similar in the case of almost $h\alpha$-*Lindelöf*.

Theorem 6.9. Let a mapping $\xi:(Z,\tau)\to(W,\Psi)$ be $h\alpha$-*irresolute*. Suppose that A is almost $h\alpha$-*compact* (*resp. almost* $h\alpha$-*Lindelöf*) subset of X. Then $f(A)$ is almost $h\alpha$-*compact* (*resp. almost* $h\alpha$-*Lindelöf*).

Proof. Suppose that $\{G_i : i\in I\}$ is $h\alpha$-*open* cover of $f(A)$. Then $f(A)\subseteq\cup\{G_i : i\in I\}$. Now $A\subseteq\cup\{f^{-1}(G_i) : i\in I\}$. Since f is $h\alpha$-*irresolute*, then $\{f^{-1}(G_i) : i\in I\}$ is an $h\alpha$-*open* cover of A. By hypothesis, A is almost $h\alpha$-*compact*, then there exists a finite subset I_0 of I such that $A\subseteq\cup\{h\alpha\text{-}Cl[f^{-1}(G_i)] : i\in I_0\}$. Since f is $h\alpha$-*irresolute*, then $h\alpha\text{-}Cl(f^{-1}(G_i))\subseteq f^{-1}[h\alpha\text{-}Cl(G_i)]$, for all $i\in I_0$. Hence it follows that $f(A)\subseteq\cup_{i\in I_0} f[f^{-1}(h\alpha\text{-}Cl(G_i))]\subseteq\cup_{i\in I_0} h\alpha\text{-}Cl(G_i)$, which implies that $f(A)\subseteq\cup_{i\in I_0} h\alpha\text{-}Cl(G_i)$. Thus $f(A)$ is almost $h\alpha$-*compact*.

The proof is similar in the case of almost $h\alpha$-*Lindelöf*.

Theorem 6.10. Let $\xi:(Z,\tau)\to(W,\Psi)$ be an $h\alpha$-*open* bijective mapping and (W,Ψ) is almost $h\alpha$-*compact*. Then (X,τ) is almost compact.

Proof. Let $\{G_i : i\in I\}$ be an open cover of X. Then $f(X)=f(\cup_{i\in I} G_i)$. Therefore $Y=\cup_{i\in I} f(G_i)$. Now, Y is almost $h\alpha$-*compact*, then there exists a finite subset I_0 of I such that $Y=\cup_{i\in I_0} h\alpha\text{-}Cl[f(G_i)]$. Since f is $h\alpha$-*open* bijective mapping, then f is $h\alpha$-*closed* mapping. Therefore, we have $h\alpha\text{-}Cl[f(G_i)]\subseteq f[Cl(G_i)]$, for all $i\in I_0$. Thus $Y\subseteq\cup_{i\in I_0} f[Cl(G_i)]\subseteq f[\cup_{i\in I_0} Cl(G_i)]$, which implies that $X=f^{-1}(Y)\subseteq\cup_{i\in I_0} Cl(G_i)$. Thus $X=\cup_{i\in I_0} Cl(G_i)$. Hence X is almost compact.

Theorem 6.11. If every collection of $h\alpha$-*closed* subsets of (Z,τ) satisfying the finite (resp. countable) intersection property, has, itself, a non-empty intersection, then X is almost $h\alpha$-*compact* (*resp. almost* $h\alpha$-*Lindelöf*).

Proof. Let $\{G_i : i\in I\}$ be an $h\alpha$-*open* cover of X. Suppose $\{G_i : i\in I\}$ has no finite sub-collection such that the $h\alpha$-*closure* of whose members cover X. Then $X-\cup_{i=1}^{i=n} h\alpha\text{-}Cl(G_i)\neq\phi$, for any $n\in N$. Therefore $X-\cup_{i=1}^{i=n} G_i\neq\phi$. Now, $\cap_{i=1}^{n} G_i^c\neq\phi$ implies $\{G_i^c : i\in I\}$ is a collection of $h\alpha$-*closed* subsets of X which has the finite intersection property. Thus $\cap_{i\in I} G_i^c\neq\phi$ implies $X\neq\cup_{i\in I} G_i$. But this is a contradiction. Hence X is almost $h\alpha$-*compact*.

A similar proof is given in the case of *almost* $h\alpha$-*Lindelöf*.

7. MILDLY H_α-COMPACT SPACES

Definition 7.1. A topological space (Z,τ) is called mildly $h\alpha$-*compact* (*resp. mildly* $h\alpha$-*Lindelöf*) provided that every $h\alpha$-*clopen* cover of X has a finite (resp. countable) subcover.

Definition 7.2. Every mildly $h\alpha$-*compact* space is mildly $h\alpha$-*Lindelöf*.

Proof. It is straightforward.

Theorem 7.3. Every almost $h\alpha$-*compact* (*resp. almost* $h\alpha$-*Lindelöf*) space (X, τ) is mildly $h\alpha$-*compact* (*resp. mildly* $h\alpha$-*Lindelöf*).

Proof. Let $\Lambda = \{H_i : i \in I\}$ be an $h\alpha$-*clopen* cover of (X, τ). Since (X, τ) is almost $h\alpha$-*compact*, then there exists a finite subset I_0 of I such that $X = \mathrm{U}_{i\in I_0} h\alpha\text{-}Cl(H_i)$. Now, $h\alpha\text{-}Cl(H_i) = H_i$. Thus (X, τ) is mildly $h\alpha$-*compact*.

A similar proof is given when (X, τ) is *almost* $h\alpha$-*Lindelöf*.

Corollary 7.4. Every $h\alpha$-*compact* (*resp.* $h\alpha$-*Lindelöf*) space is mildly $h\alpha$-*compact* (*resp. mildly* $h\alpha$-*Lindelöf*).

Theorem 7.5. If F is an $h\alpha$-*clopen* subset of a mildly $h\alpha$-*compact* (*resp. mildly* $h\alpha$-*Lindelöf*) space X, then F is mildly $h\alpha$-*compact* (*resp. mildly* $h\alpha$-*Lindelöf*).

Proof. Let F be an $h\alpha$-*clopen* subset of X and $\{G_i : i \in I\}$ be an $h\alpha$-*clopen* cover of F. Then F^c is an $h\alpha$-*clopen* set and $F \subseteq \mathrm{U}_{i\in I} G_i$. Therefore $X = (\mathrm{U}_{i\in I} G_i) \mathrm{U} F^c$. Since X is mildly $h\alpha$-*compact*, then there exists a finite subset I_0 of I such that $X = (\mathrm{U}_{i\in I_0} G_i) \mathrm{U} F^c$. So $F \subseteq (\mathrm{U}_{i\in I_0} G_i)$. Hence F is mildly $h\alpha$-*compact*.

The proof is similar in a case of mildly $h\alpha$-*Lindelöf*.

Theorem 7.6. If A is a mildly $h\alpha$-*compact* (*resp. mildly* $h\alpha$-*Lindelöf*) subset of X and B is an $h\alpha$-*clopen* subset of X, then $A \mathbf{I} B$ is mildly $h\alpha$-*compact* (*resp. mildly* $h\alpha$-*Lindelöf*).

Proof. Let $\Lambda = \{G_i : i \in I\}$ be an $h\alpha$-*clopen* cover of $A \mathbf{I} B$. Then $A \subseteq (\mathrm{U}_{i\in I} G_i) \mathrm{U} B^c$. Since A is mildly $h\alpha$-*compact*, then there exists a finite subset I_0 of I such that $A \subseteq (\mathrm{U}_{i\in I_0} G_i) \mathrm{U} B^c$. Therefore $A \mathrm{I} B \subseteq \bigcup_{i\in I_0} G_i$. Thus $A \mathbf{I} B$ is mildly $h\alpha$-*compact*.

The proof is similar in case of mildly $h\alpha$-*Lindelöf*.

Theorem 7.7. If $\xi : (Z, \mathrm{T}) \to (W, \Psi)$ is an $h\alpha$-*open* bijective mapping and (W, Ψ) is mildly $h\alpha$-*compact*, then (Z, T) is mildly compact.

Proof. Let $\{G_i : i \in I\}$ be a clopen cover for X. Then $f(X) = f(\mathrm{U}_{i\in I} G_i)$. Hence $Y = \mathrm{U}_{i\in I} f(G_i)$. Since f is $h\alpha$-*open* bijective mapping, then f is $h\alpha$-*closed*. Therefore $\{f(G_i) : i \in I\}$ is an $h\alpha$-*clopen* cover of X. Since Y is mildly $h\alpha$-*compact*, then there exists a finite subset I_0 of I such that $Y = \mathrm{U}_{i\in I_0} f(G_i)$. Therefore $X = \mathrm{U}_{i\in I_0} G_i$. Thus X is mildly compact.

Proposition 7.8. A subset A of (Z, T) is mildly compact (*resp. mildly Lindelöf*) if and only if (Z, T_A) is mildly compact (*resp. mildly Lindelöf*).

8. CONCLUSIONS

We have used $h\alpha$-*open* sets to introduce the new concepts of notions in topological spaces, namely $h\alpha$-*compact* space, countably $h\alpha$-*compact* space, $h\alpha$-*Lindelöf* space, almost $h\alpha$-*compact* space, and mildly $h\alpha$-*compact* space and have investigated several properties and the characterization of these new concepts.

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