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Semi-Star-Alpha-Irresolute Topological Vector Spaces

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ABSTRACT

We introduce and study a new class of spaces, namely *semi*α-irresolute* topological vector spaces via *semi*α-open* sets. We also introduce a new form of topological vector space called contra *semi*α* topological vector space. We explore and investigate several properties and characterizations of these new notions of *semi*α-irresolute* topological vector space and contra *semi*α* topological vector space. Furthermore, we define and investigate new separation axioms, namely $s^*\alpha$-T_i-spaces $(i = 0, 1, 2, 3, 4)$ and give complete descriptions for each one of them. Lastly, we introduce a new concept of semi*α-limit points of a set and study main properties, particularly on the spaces that possess the difference property.

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Key Words—Topological space, Semi*α-open set, Semi*α-closed set, *Semi*α-irresolute* topological vector space, Contra *semi*α-continuous* function, Contra *Semi*α* topological vector space, semi*α-T_i-space $(i = 0, 1, 2, 3, 4)$, Semi*α-limit poiint,

1. INTRODUCTION

The concept of topological vector spaces was first given by Kolmogroff [6] in 1934. Since then, different researchers have explored many interesting and useful properties of topological vector spaces. Due to nice properties, topological vector spaces earn a great importance and remain a topological notion in fixed point theory, operator theory, rational inequalities, etc. Now a days the researchers not only make use of topological vector spaces in other fields for developing new notions but also extend the concept of topological vector spaces with every possible way, making this field of study a more convenient and understandable. Recently, M. Khan et al. [4] introduced the notion of *s-topological* vector space, a generalization of topological vector spaces. In 2016, Khan and Iqbal [5] introduced irresolute topological vector spaces which are particular types of *s-topological* vector spaces, but irresolute topological vector spaces are independent of topological vector spaces. Ibrahim [3] initiated the study of *α-topological* vector spaces. Ram et al. [12] introduced the notion of almost *s-topological* vector spaces. In 2018, Radhi I. M. Ali, et al. [1] introduced *semi – pre* irresolute topological vector spaces. The study of generalized closed (*briefly g-closed*) sets in topological spaces was initiated by N. Levine [9] in 1970. Quite recently, Robert and Missier [13, 14, 15, 16, 17] introduced some new concepts, namely *semi*α-open* sets, *semi*α-closed* sets, the *semi*α-closure, semi*α-derived* set and *semi*α-frontier* of a subset. We introduce *semi*α* irresolute topological vector space and contra *semi*α* topological vector space by using *semi*α-open* sets and investigate several properties and characterizations of these notions of *semi*α-irresolute* topological vector space and contra *semi*α* topological vector space. Furthermore, we define and investigate new separation axioms, namely $s^*\alpha$-T_i-spaces $(i = 0, 1, 2, 3, 4)$ and give complete descriptions for each one of them. In the last section, we explore a new concept of *semi*α-limit* points of a set and initiate new types of separation axioms using *semi*α-open* sets and illustrate the relationships between them.

2. PRELIMINARIES

A subset A of a topological space (X, τ) is said to be open if $A \in \tau$. A subset A of a topological space X is said to be closed if the set $X - A$ is open. The interior of a subset A of a topological space X is defined as the union of all open sets contained in A. It is denoted by $Int(A)$. The closure of a subset A of a topological space X is defined as the intersection of all closed sets containing A. It is denoted by $Cl(A)$. Throughout this paper, (X, τ) represent the *non-empty* topological spaces on which no separation axioms are assumed, unless otherwise mentioned.

Definition 2.1. A subset A of a topological space (X, τ) is said to be *generalized* closed (*briefly g-closed*) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

Definition 2.2. A subset A of a topological space (X, τ) is said to be *generalized* open (*briefly g-open*) if $X - A$ is *g-closed* in X.

Definition 2.3. Let A be a subset of a space X. The generalized closure of A is defined as the intersection of all *g-closed* sets containing A and is denoted by $Cl^*(A)$.

Definition 2.4. The union of all *generalized* open sets containing a set A is called the *generalized-interior* of A and is denoted by $Int^*(A)$.

Definition 2.5. A subset A of a topological space (X, τ) is called α*-open* if $A \subseteq Int\left[Cl\left(Int(A)\right)\right]$.

Definition 2.6. The union of all α*-open* sets containing a set A is called α*-interior* of A and is denoted by $\alpha Int(A)$.

Definition 2.7. A subset A of a topological space (X, τ) is called α*-closed* if $Cl\left[Int\left(Cl(A)\right)\right] \subseteq A$.

Definition 2.8. Let A be a subset of a topological space (X, τ). Then the α*-closure* of A is defined as the intersection of all α*-closed* sets containing A and is denoted by $\alpha Cl(A)$.

Definition 2.9. [15] A subset A of a topological space (X,τ) is called *semi*α-open* if $A \subseteq Cl^*\left[\alpha Int(A)\right]$.

The family of all *semi*α-open* subsets of the space X is denoted by $s^*\alpha\text{-}O(X)$.

Definition 2.10. [15] The union of all *semi*α-open* sets containing a set A is called *semi*α-interior* of A and is denoted by $s^*\alpha\text{-}Int(A)$.

Definition 2.11. [16] A subset A of a topological space (X, τ) is called *semi*α-closed* if $A \subseteq Int^*\left[\alpha Cl(A)\right]$.

The family of all *semi*α-closed* subsets of the space X is denoted by $s^*\alpha\text{-}C(X)$.

Definition 2.12. [16] Let A be a subset of a topological space (X,τ). Then the *semi*α-closure* of A is defined as the intersection of all *semi*α-closed* sets in X containing A and is denoted by $s^*\alpha\text{-}Cl(A)$.

Definition 2.13. A subset A of a topological space (X, τ) is called *semi-open* if $A \subseteq Cl\left[Int(A)\right]$.

Remark 2.14. [17] Let A be a subset of a topological space (X, τ). Then the following implications are true:

open set $\Rightarrow$ α*-open set* $\Rightarrow$ *semi*α-open set* $\Rightarrow$ *semi-open set*

Theorem 2.15. [15] Let (X,τ) be a topological space. Then arbitrary union of *semi*α-open* sets in X is also *semi*α-open.*

Theorem 2.16. [15] Let A be a *semi*α-open* set in X and B be an open set in X. Then $A\cap B$ is *semi*α-open* in X.

Theorem 2.17. [15] Let (X,τ) be a topological space and A be a subset of X. Then A is *semi*α-open* if and only if $s^*\alpha\text{-}Int(A)=A.$

Theorem 2.18. [15] Let A be a subset of a topological space (X,τ). Then the following assertions are true:

(1) A is *semi*α-open* in X.

(2) $A\subseteq Cl^*\left[\alpha Int(A)\right].$

(3) $Cl^*\left[\alpha Int(A)\right]=Cl^*(A).$

Theorem 2.19. [16] Let A be a subset of a topological space (X,τ). Then the following assertions are true:

(1) A is *semi*α-closed* in X.

(2) $Int^*\left[\alpha Cl(A)\right]\subseteq A.$

(3) $Int^*\left[\alpha Cl(A)\right]=Int^*(A).$

Theorem 2.20. [15] Let (X,τ) be a topological space. Let $A\subseteq X$ and $x\in X.$ Then $x\in s^*\alpha\text{-}Int(A)$ if and only if there exists *semi*α-open* set U in X such that $x\in U\subseteq A.$

Theorem 2.21. [16] Let (X,τ) be a topological space and A be a subset of X. Then A is *semi*α-closed* if and only if $s^*\alpha\text{-}Cl(A)=A.$

Theorem 2.22. [16] Let (X,τ) be a topological space. Let $A\subseteq X$ and $x\in X$. Then $x\in s^*\alpha\text{-}Cl(A)$ if and only if any *semi*α-open* set U in X containing x intersects $A.$

Proposition 2.23. A subset A of a topological space (X,τ) is *semi*α-open* in X if and only if $X-A$ is *semi*α-closed* in X.

Definition 2.24. A mapping $f:(X,\tau)\to(Y,\sigma)$ is called *semi*α-irresolute* at a point $x\in X$ if for each *semi*α-open* subset V in Y containing $f(x)$, there is a *semi*α-open* subset U of X such that $f(U)$ is a subset of V.

Theorem 2.25. Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then the following statements are equivalent:

(1) f is semi*α-irresolute.

(2) $f\left[s^*\alpha\text{-}Cl(A)\right]\subseteq s^*\alpha\text{-}Cl\left[f(A)\right]$ holds for every subset A of (X,τ).

(3) $f^{-1}(B)$ is *semi*α-closed* in (X,τ), for every semi*α-closed set B of (Y,σ).

(4) $f^{-1}(B)$ is *semi*α-open* in (X,τ), for every semi*α-open set B of (Y,σ).

Proof. $(1)\Rightarrow(2)$: Let $y\in f\left[s^*\alpha\text{-}Cl(A)\right]$ and V be any *semi*α-open* set containing y. Then there exists a point $x\in X$ and a *semi*α-open* open set U such that $f(x)=y,\ x\in U$ and $f(U)\subseteq V$. Since $x\in s^*\alpha\text{-}Cl(A)$, we have $U\cap A\neq\phi$ and hence $\phi\neq f(U\cap A)\subseteq f(U)\cap f(A)\subseteq V\cap f(A)$. This implies $y\in s^*\alpha\text{-}Cl\left[f(A)\right]$. Therefore we have $f\left[s^*\alpha\text{-}Cl(A)\right]\subseteq s^*\alpha\text{-}Cl\left[f(A)\right]$.

$(2)\Rightarrow(3)$: Let B be a *semi*α-closed* set in (Y,σ). Therefore $s^*\alpha\text{-}Cl(B)=B$. By using (2) $f\left[s^*\alpha\text{-}Cl\left(f^{-1}(B)\right)\right]\subseteq s^*\alpha\text{-}Cl(B)=B$. Therefore we have $s^*\alpha\text{-}Cl\left(f^{-1}(B)\right)\subseteq f^{-1}(B)$. Hence $f^{-1}(B)$ is *semi*α-closed*.

$(3)\Rightarrow(4)\Rightarrow(1)$: Easy.

3. PROPERTIES OF SEMI-STAR-ALPHA-IRRESOLUTE TOPOLOGICAL VECTOR SPACES

In this section, we define and investigate basic properties of *semi*α-irresolute* topological vector spaces.

Definition 3.1. A topological vector space $\left(X_{(K)},\tau\right)$ is called *semi*α-irresolute* topological vector space $(s^*\alpha ITV)$ whenever the following conditions are satisfied.

(1) for each $x,y\in X$ and for each *semi*α-open* neighborhood W of $x+y$ in X, there exist *semi*α-open* neighborhoods U and V in X of x and y respectively, such that $U+V\subseteq W$.

(2) for each $x\in X,\ \lambda\in K$ and for each *semi*α-open* neighborhood W of λx in X, there exist *semi*α-open* neighborhoods U of λ in K and V of x in X, such that $U.V\subseteq W$.

Theorem 3.2. Let $\left(X_{(K)},\tau\right)$ be a *semi*α-irresolute* topological vector space. Then the following assertions hold.

(1) The (left) right translation $T_x:X\to X$ defined by $T_x(y)=y+x;$ for all $x,y\in X$, is *semi*α-irresolute.*

(2) The translation $M_\lambda:X\to X$, defined by $M_\lambda(x)=\lambda x$, for all $x\in X$, is *semi*α-irresolute.*

Proof. (1) Let W be a *semi*α-open* neighborhood of $T_x(y)=x+y$. Then by definition, there exist *semi*α-open* neighborhoods U and V in X containing y and x respectively, such that $U+V\subseteq W$. This gives that $T_x(U)=U+x\subseteq U+V\subseteq W$. This proves that, $T_x:X\to X$ is *semi*α-irresolute* mapping.

(2) Let $x\in X,\ \lambda\in K$. Then $M_\lambda(x)=\lambda x$. Let W be a *semi*α-open* neighborhood of λx. Then by definition of *semi*α-irresolute* topological vector space, there exist *semi*α-open* neighborhoods U in K of λ and V in X of x, such that $U.V\subseteq W$. This gives that $M_\lambda(V)=\lambda V\subseteq U.V\subseteq W$. This proves that is *semi*α-irresolute* mapping.

Theorem 3.3. Let $\left(X_{(K)},\tau\right)$ be a *semi*α-irresolute* topological vector space. Let $A\in s^*\alpha\text{-}O(X)$. Then the following statements are true.

(1) $x+A\in s^*\alpha\text{-}O(X)$, for every $x\in X$.

(2) $\lambda A\in s^*\alpha\text{-}O(X)$, for every *non-zero* scalar $\lambda\in K$.

Proof. (1) $y\in x+A$. Then $y=x+a$ for some $a\in A$. By definition of *semi*α-irresolute* topological vector space, there exist *semi*α-open* sets U and V in X containing $-x$ and y respectively such that $U+V\subseteq A$. This gives

$-x+V \subseteq U+V \subseteq A$. This implies $y \in V \subseteq x+A$. Therefore $y \in s^*\alpha\text{-}Int(x+A)$. Hence, $x+A = s^*\alpha\text{-}Int(x+A)$. This proves that $x+A$ is *semi*α-open* in X.

(2) Let $x \in \lambda A$. Then $x = \lambda a$ for some $a \in A$. Thus $a = \frac{1}{\lambda}x \in A$. By the definition of *semi*α-irresolute* topological vector space, there exist *semi*α-open* sets U in K containing $\frac{1}{\lambda}$ and V in X containing x such that $U.V \subseteq A$. This implies that $a = \frac{1}{\lambda}x \in \frac{1}{\lambda}V \subseteq U.V \subseteq A$. Hence $x \in V \subseteq \lambda A$. Thus we obtain $x \in s^*\alpha\text{-}Int(\lambda A)$. Therefore, it follows that $\lambda A \subseteq s^*\alpha\text{-}Int(\lambda A)$. Hence, $\lambda A = s^*\alpha\text{-}Int(\lambda A)$. This shows that $\lambda A \in s^*\alpha\text{-}O(X)$.

Corollary 3.4. Let $(X_{(K)}, \tau)$ be a *semi*α-irresolute* topological vector space. Let A be a *semi*α-open* subset of X. Then the following statements are true.

(1) $x+A \subseteq s^*\alpha\text{-}Cl[s^*\alpha\text{-}Int(x+A)]$ for each $x \in X$.

(2) $\lambda A \subseteq s^*\alpha\text{-}Cl[s^*\alpha\text{-}Int(\lambda A)]$ for any non-zero scalar λ.

Theorem 3.5. Let $(X_{(K)}, \tau)$ be a *semi*α-irresolute* topological vector space. Let $A \in s^*\alpha\text{-}O(X)$ and B be any subset of X. Then prove $A+B \in s^*\alpha\text{-}O(X)$.

Proof. Suppose $A \in s^*\alpha\text{-}O(X)$ and $B \subseteq X$. Then, for each $b \in B$ and by Theorem 3.3 (1), we have $A+b \in s^*\alpha\text{-}O(X)$. Then $A+B = \bigcup\{A+b : b \in B\}$ is a union of *semi*α-open* sets. Since arbitrary union of *semi*α-open* sets is *semi*α-open,* therefore $A+B$ is *semi*α-open* in X.

Corollary 3.6. Suppose $(X_{(K)}, \tau)$ is *semi*α-irresolute* topological vector space and let $A \in s^*\alpha\text{-}O(X)$. Then the set $U = \bigcup_{n=1}^{\infty}(nA)$ is *semi*α-open* in X.

Theorem 3.7. Let $(X_{(K)}, \tau)$ be a *semi*α-irresolute* topological vector space. Let $A \subseteq X$. Then

(1) $s^*\alpha\text{-}Int(x+A) = x+s^*\alpha\text{-}Int(A)$ for any $x \in X$.

(2) $s^*\alpha\text{-}Int(\lambda A) = \lambda[s^*\alpha\text{-}Int(A)]$ for non-zero scalar $\lambda \in K$.

Proof. (1) By applying Theorem 3.3 (1), $x+s^*\alpha\text{-}Int(A)$ is *semi*α-open.* Therefore $x+s^*\alpha\text{-}Int(A) \subseteq x+A$ implies $x+s^*\alpha\text{-}Int(A) \subseteq s^*\alpha\text{-}Int(x+A)$. Now suppose that $z \in s^*\alpha\text{-}Int(x+A)$. Then $z = x+y$ for some $y \in A$. By the definition of *semi*α-irresolute* topological vector space, there exist *semi*α-open* sets U and V in X containing x and y respectively, such that $U+V \subseteq x+s^*\alpha\text{-}Int(A)$. This gives that $z = x+y \in x+V \subseteq U+V \subseteq x+s^*\alpha\text{-}Int(A)$. Therefore it follows that $s^*\alpha\text{-}Int(x+A) \subseteq x+s^*\alpha\text{-}Int(A)$.

(2) By Theorem 3.3(2), $\lambda[s^*\alpha\text{-}Int(A)]$ is *semi*α-open.* Thus $\lambda[s^*\alpha\text{-}Int(A)] \subseteq \lambda A$ implies that $\lambda[s^*\alpha\text{-}Int(A)] \subseteq s^*\alpha\text{-}Int(\lambda A)$. Next, if $y \in s^*\alpha\text{-}Int(\lambda A)$, then $y = \lambda x$ for some $x \in A$. By the definition of *semi*α-irresolute* topological vector space, there exist *semi*α-open* sets U of λ and V of x in X such that $U.V \subseteq s^*\alpha\text{-}Int(\lambda A) \subseteq \lambda A$. So $y = \lambda x \in \lambda V \subseteq U.V \subseteq \lambda A$. This implies that $x \in V \subseteq A$ and since V is *semi*α-open.* Thus it shows that $x \in s^*\alpha\text{-}Int(A)$. Consequently, $y = \lambda x \in \lambda[s^*\alpha\text{-}Int(A)]$. Hence, we conclude that $s^*\alpha\text{-}Int(\lambda A) = \lambda[s^*\alpha\text{-}Int(A)]$.

Theorem 3.8. Let F be any *semi*α-closed* subset of a *semi*α-irresolute* topological vector space X. Then the following assertions are true:

(1) $x+F \in s^*\alpha\text{-}C(X)$, for every $x \in X$.

(2) $\lambda F \in s^*\alpha\text{-}C(X)$, for each *non-zero* scalar $\lambda \in K$.

Proof. (1) Suppose that $y \in s^*\alpha\text{-}Cl(x+F)$. Consider $z=-x+y$ and let W be any *semi*α-open* set in X containing z. Then there exist *semi*α-open* sets U and V in X such that $-x \in U$, $y \in V$ and $U+V \subseteq W$. Since $y \in s^*\alpha\text{-}Cl(x+F)$, $(x+F)\cap V \neq \phi$. So, there is $a \in (x+F)\cap V$. Thus $a \in x+F$ and $a \in V$. Hence $-x+a \in F$ and $-x+a \in U+V$. Therefore $-x+a \in F \cap (U+V) \subseteq F \cap W$. Thus $F \cap W \neq \phi$. Therefore $z=-x+y \in s^*\alpha\text{-}Cl(F)=F$. Hence $y \in x+F$. Thus $x+F=s^*\alpha\text{-}Cl(x+F)$. This proves that $x+F$ is *semi*α-closed* set in X.

(2) Assume that $x \in s^*\alpha\text{-}Cl(\lambda F)$. Let W be any *semi*α-open* neighborhood of $y=\frac{1}{\lambda}x$ in X. Since X is *semi*α-irresolute* topological vector space, there exist *semi*α-open* sets U in K containing $\frac{1}{\lambda}$ and V in X containing x such that $U.V \subseteq W$. By hypothesis, $(\lambda F)\cap V \neq \phi$. Therefore, there is $a \in (\lambda F)\cap V$. Thus $a \in \lambda F$ and $a \in V$. Hence $\frac{1}{\lambda}a \in F$ and $\frac{1}{\lambda}a \in \frac{1}{\lambda}V \subseteq U.V \subseteq W$. Therefore $F \cap W \neq \phi$. Hence $y=\frac{1}{\lambda}x \in s^*\alpha\text{-}Cl(F)=F$. Thus $x \in \lambda F$ and thereby, $\lambda F=s^*\alpha\text{-}Cl(\lambda F)$. Hence $\lambda F \in s^*\alpha\text{-}C(X)$.

Corollary 3.9. Let $(X_{(K)},\tau)$ be a *semi*α-irresolute* topological vector space. Let $A \subseteq X$. Then the following statement is true.

$s^*\alpha\text{-}Cl[x+s^*\alpha\text{-}Cl(A)]=x+s^*\alpha\text{-}Cl(A)$ for each $x \in X$.

Theorem 3.10. Let $(X_{(K)},\tau)$ be a *semi*α-irresolute* topological vector space and S be a subspace of X. If S contains a *non-empty* *semi*α-open* subset of X, then S is *semi*α-open* set in X.

Proof. Suppose U is a *non-empty* *semi*α-open* subset of X such that $U \subseteq S$. By Theorem 3.3 (1), $T_x(U)=U+y$ is *semi*α-open* subset of X. Thus $S=\cup\{U+y : y \in S\}$ is *semi*α-open* in X being union of *semi*α-open* sets.

Theorem 3.11. Let A be any subset of a *semi*α-irresolute* topological vector space X. Then prove the following statements.

(1) $x+s^*\alpha\text{-}Cl(A)=s^*\alpha\text{-}Cl(x+A)$ for any $x \in X$.

(2) $s^*\alpha\text{-}Cl(\lambda A)=\lambda[s^*\alpha\text{-}Cl(A)]$ for any *non-zero* scalar λ.

Proof. (1) By applying Theorem 3.8(1), $x+s^*\alpha\text{-}Cl(A)$ is *semi*α-closed*. Hence $x+A \subseteq s^*\alpha\text{-}Cl(x+A)$ implies $s^*\alpha\text{-}Cl(x+A) \subseteq x+s^*\alpha\text{-}Cl(A)$. For the reverse inclusion, let $z \in x+s^*\alpha\text{-}Cl(A)$. Then $z=x+y$, for some $y \in s^*\alpha\text{-}Cl(A)$. Let W be any *semi*α-open* neighborhood of z in X. Then, there exist *semi*α-open* neighborhoods U and V of x and y respectively in X such that $U+V \subseteq W$. Since $y \in s^*\alpha\text{-}Cl(A)$, $A\cap V \neq \phi$. Consider $a \in A\cap V$. Then $x+a \in (x+A)\cap(U+V) \subseteq (x+A)\cap W \Rightarrow (x+A)\cap W \neq \phi$. Consequently, $z \in s^*\alpha\text{-}Cl(x+A)$. So $x+s^*\alpha\text{-}Cl(A) \subseteq s^*\alpha\text{-}Cl(x+A)$. Hence, $x+s^*\alpha\text{-}Cl(A)= s^*\alpha\text{-}Cl(x+A)$

(2) By Theorem 3.8 (2), $\lambda[s^*\alpha\text{-}Cl(A)]$ is *semi*α-closed*. Therefore $\lambda A \subseteq \lambda[s^*\alpha\text{-}Cl(A)]$ implies that $s^*\alpha\text{-}Cl(\lambda A) \subseteq \lambda[s^*\alpha\text{-}Cl(A)]$. Next, suppose that $y = \lambda x \in \lambda[s^*\alpha\text{-}Cl(A)]$, where $x \in s^*\alpha\text{-}Cl(A)$ and let W be any *semi*α-open* neighborhood of $y = \lambda x$ in X. Then we get *semi*α-open* sets U in K containing λ and V in X containing x such that $U.V \subseteq W$. Since $x \in s^*\alpha\text{-}Cl(A)$, there is $a \in A \cap V$ and hence $a \in A$ and $a \in V$. Then $\lambda a \in \lambda A$ and $\lambda a \in \lambda V \subseteq U.V \subseteq W$. Thus $\lambda a \in (\lambda A) \cap W$ and hence $(\lambda A) \cap W \neq \phi$. This implies that $y = \lambda x \in s^*\alpha\text{-}Cl(\lambda A)$. Then this shows that $\lambda[s^*\alpha\text{-}Cl(A)] \subseteq s^*\alpha\text{-}Cl(\lambda A)$. Consequently, we obtain $s^*\alpha\text{-}Cl(\lambda A) = \lambda[s^*\alpha\text{-}Cl(A)]$.

Theorem 3.12. Let $(X_{(K)}, \tau)$ be a *semi*α-irresolute* topological vector space. Let A and B be subsets of X. Then $s^*\alpha\text{-}Cl(A) + s^*\alpha\text{-}Cl(B) \subseteq s^*\alpha\text{-}Cl(A+B)$.

Proof. Let $x \in s^*\alpha\text{-}Cl(A)$ and $y \in s^*\alpha\text{-}Cl(B)$, let W be a *semi*α-open* neighborhood of $x+y$. Then there exist *semi*α-open* neighborhoods U and V of x and y respectively, such that $U+V \subseteq W$. Since, $x \in s^*\alpha\text{-}Cl(A)$, $y \in s^*\alpha\text{-}Cl(B)$, there are $a \in A \cap U$ and $b \in B \cap V$. Then, $a+b \in (A+B) \cap (U+V) \subseteq (A+B) \cap W$. This implies that $x+y \in s^*\alpha\text{-}Cl(A+B)$. Therefore we conclude that $s^*\alpha\text{-}Cl(A) + s^*\alpha\text{-}Cl(B) \subseteq s^*\alpha\text{-}Cl(A+B)$.

Theorem 3.13. Let $(X_{(K)}, \tau)$ be a *semi*α-irresolute* topological vector space. For given $y \in X$ and $\lambda \in K$ with $\lambda \neq 0$, each translation mapping $T_y : X \to X$ defined by $T_y(x) = x + y$ and multiplication mapping $M_\lambda : X \to X$ defined by $M_\lambda(x) = \lambda x$, where $x \in X$, is *semi*α-homeomorphism* onto itself.

Proof. First, we show that $T_y : X \to X$ is *semi*α-homeomorphism*. It is obviously bijective. By Theorem 3.2 (1), T_y is *semi*α-irresolute*. Moreover, T_y is *pre-semi*α-open* because for any *semi*α-open* set U, by Theorem 3.3 (1), $T_y(U) = U + y$ is *semi*α-open*. Similarly, we can prove that M_λ is *semi*α-homeomorphism*.

Theorem 3.14. Let $(X_{(K)}, \tau)$ be a *semi*α-irresolute* topological vector space. Then any *semi*α-open* subspace of X is *semi*α-closed* in X.

Proof. Let G be a *semi*α-open* subspace of X. Then by Theorem 3.3 (1), for any $x \in X - G$, $G + x$ is *semi*α-open*. We also clearly have $x \in G + x \subseteq X - G$. Then, $Z = \bigcup\{G + x : x \in X - G\} = X - G$ being a union of *semi*α-open* sets is *semi*α-open*. Therefore, $G = X - Z$ is *semi*α-closed*.

Theorem 3.15. Let $(X_{(K)}, \tau)$ be a *semi*α-irresolute* topological vector space and B be a *semi*α-open* set in X. Then for any subset A of X, we have $A + B = s^*\alpha\text{-}Cl(A) + B$.

Proof. Since we know that $A \subseteq s^*\alpha\text{-}Cl(A)$, so $A + B \subseteq s^*\alpha\text{-}Cl(A) + B$. Conversely, let $y \in s^*\alpha\text{-}Cl(A) + B$ and write $y = x + b$, where $x \in s^*\alpha\text{-}Cl(A)$ and $b \in B$. There exists a *semi*α-open* neighborhood V of zero such that $T_b(V) = V + b \subseteq B$. Now, V is *semi*α-open* neighborhood of 0 in X, this gives that $-V$ is also *semi*α-open* neighborhood of 0 in X. Then $x - V$ is a *semi*α-open* neighborhood of x. Since $x \in s^*\alpha\text{-}Cl(A)$, so there exists an element $a \in A \cap (x - V)$. We know that $y = x + b = a - a + x + b \in a + V + b \subseteq A + B$. Therefore, $s^*\alpha\text{-}Cl(A) + B \subseteq A + B$. Hence, consequently, we obtain $A + B = s^*\alpha\text{-}Cl(A) + B$.

Theorem 3.16. Let $(X_{(K)}, \tau)$ be a *semi*α-irresolute* topological vector space. Then the scalar multiple of *semi*α-closed* set is *semi*α-closed*.

Proof. Let B be a *semi*$^*\alpha$-closed* set in X and let $\lambda \in K-\{0\}$. Then $X-B$ is *semi*$^*\alpha$-open* set in X. Now $M_\lambda(X-B)=\lambda(X-B)=\lambda X-\lambda B=$

$X-\lambda B \in s^*\alpha\text{-}O(X)$. Therefore, $\lambda B \in s^*\alpha\text{-}C(X)$.

Definition 3.17. A topological space (X,τ) is said to be *semi*$^*\alpha$-compact* if every cover of X by *semi*$^*\alpha$-open* sets of X has a finite sub cover. A subset A of X is said to be *semi*$^*\alpha$-compact* relative to X if every cover of A by *semi*$^*\alpha$-open* sets of X has a finite sub cover.

Theorem 3.18. Let $(X_{(K)},\tau)$ be a *semi*$^*\alpha$-irresolute* topological vector space and let A be any *semi*$^*\alpha$-compact* set in X. Then prove that $x+A$ is *semi*$^*\alpha$-compact* for each $x \in X$.

Proof. Let $\Omega=\{U_\alpha : \alpha \in \Lambda\}$ be a *semi*$^*\alpha$-open* cover of $x+A$. Then $A \subseteq \cup\{-x+U_\alpha : \alpha \in \Lambda\}$ and $\{-x+U_\alpha : \alpha \in \Lambda\} \subseteq s^*\alpha\text{-}O(X)$. By hypothesis, $A \subseteq \cup\{-x+U_\alpha : \alpha \in \Lambda_0\}$ for some finite subset $\Lambda_0 \subseteq \Lambda$. Whence we find that $x+A \subseteq \cup\{U_\alpha : \alpha \in \Lambda_0\}$. This shows that $x+A$ is *semi*$^*\alpha$-compact*. Hence, the proof is complete.

Theorem 3.19. Let $(X_{(K)},\tau)$ be a *semi*$^*\alpha$-irresolute* topological vector space. The scalar multiple of *semi*$^*\alpha$-compact* set is *semi*$^*\alpha$-compact*.

Proof. Let A be a *semi*$^*\alpha$-compact* subset of X. If $\lambda=0$ we are nothing to prove. Assume $\lambda \in K-\{0\}$. Let $\Psi=\{U_\alpha : \alpha \in \Lambda\}$ be a *semi*$^*\alpha$-open* cover of λA. Then $\lambda A \subseteq \cup\Psi$ and this implies that $A \subseteq \left(\frac{1}{\lambda}\right)(\cup\Psi)=$ $\left(\frac{1}{\lambda}\right)(\cup\{U_\alpha : \alpha \in \Lambda\})=\cup\left\{\left(\frac{1}{\lambda}\right)U_\alpha : \alpha \in \Lambda\right\}$. Since $\{U_\alpha : \alpha \in \Lambda\} \subseteq s^*\alpha\text{-}O(X)$ and $(X_{(K)},\tau)$ is *semi*$^*\alpha$-irresolute* topological vector space. Therefore, we have $\left\{\left(\frac{1}{\lambda}\right)U_\alpha : \alpha \in \Lambda\right\} \subseteq s^*\alpha\text{-}O(X)$. By hypothesis A is *semi*$^*\alpha$-compact*, therefore there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $A \subseteq \cup\left\{\left(\frac{1}{\lambda}\right)U_\alpha : \alpha \in \Lambda_0\right\}$. This implies that $\lambda A \subseteq \cup\{U_\alpha : \alpha \in \Lambda_0\}$. Hence λA is *semi*$^*\alpha$-compact*.

Definition 3.20. A mapping $f:(X_{(K)},\tau_X)\to(Y_{(K)},\tau_Y)$ is said to be linear if $f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$, for all $x,y \in X$ and $\alpha,\beta \in K$.

Definition 3.21. A mapping $f:X\to K$ is called linear functional if $f(\alpha x+\beta y)=\alpha f(x)+\beta f(y)$, for all $x,y \in X$ and $\alpha,\beta \in K$. The kernel of f is defined by $Ker(f)=\{x \in X : f(x)=0\}$.

Theorem 3.22. Let $f:(X_{(K)},\tau_X)\to(Y_{(K)},\tau_Y)$ be a linear mapping such that f is *semi*$^*\alpha$-irresolute* at 0. Then f is *semi*$^*\alpha$-irresolute* on X.

Proof. Let x be any *non-zero* element of X and V be any *semi*$^*\alpha$-open* set in Y containing $f(x)$. Since the translation of a *semi*$^*\alpha$-open* set in a *semi*$^*\alpha$-irresolute* topological vector spaces is *semi*$^*\alpha$-open*, $V-f(x)$ is *semi*$^*\alpha$-open* set in Y containing 0. Since f is *semi*$^*\alpha$-irresolute* at 0, there exists a *semi*$^*\alpha$-open* set U in X containing 0 such that $f(U) \subseteq V-f(x)$. Furthermore, the linearity of f implies that $f(x+U) \subseteq V$. By Theorem 3.3 (1), $x+U$ is *semi*$^*\alpha$-open* and hence f is *semi*$^*\alpha$-irresolute* at x. By hypothesis, f is *semi*$^*\alpha$-irresolute* at 0.

This reflects that f is *semi*α-irresolute.*

Corollary 3.23. Let $\left(X_{(K)},\tau\right)$ be a *semi*α-irresolute* topological vector space. Let $f:X\to K$ be a linear function which is *semi*α-irresolute* at 0. Then the set $F=\{x\in X:f(x)=0\}$ is *semi*α-closed.*

4. CHARACTERIZATIONS OF SEMI-STAR-ALPHA-IRRESOLUTE TOPOLOGICAL VECTOR SPACES

In this section, we give some characterizations of *semi*α-irresolute* topological vector spaces.

Theorem 4.1. Let (X,τ) be a *semi*α-irresolute* topological vector space and $x\in X,$ Then the following assertions are true:

(1) If $U\in s^{*}\alpha\text{-}N(x),$ then $x\in U.$

(2) If $U\in s^{*}\alpha\text{-}N(x)$ and V is a neighborhood of $x,$ then $U\cap V\in s^{*}\alpha\text{-}N(x).$

(3) If $U\in s^{*}\alpha\text{-}N(x),$ then there exists $V\in s^{*}\alpha\text{-}N(x)$ such that $U\in s^{*}\alpha\text{-}N(y),$ for all $y\in V.$

(4) If $U\in s^{*}\alpha\text{-}N(x)$ and $U\subseteq V,$ then $V\in s^{*}\alpha\text{-}N(x).$

(5) If $U\in s^{*}\alpha\text{-}N(0),$ then $\lambda U\in s^{*}\alpha\text{-}N(0)$ for every non-zero element $\lambda\in R.$

(6) If $U\in s^{*}\alpha\text{-}N(x)$ and V is a *semi*α-neighborhood* of $x,$ then $U\cap V\in s^{*}\alpha\text{-}N(x).$

(7) $U\in s^{*}\alpha\text{-}N(0)$ if and only if $x+U\in s^{*}\alpha\text{-}N(x).$

Proof. We will prove (2), (5) and (7) while the proofs of others follow easily.

(2) If U is a *semi*α-neighborhood* of $x,$ and V is a neighborhood of $x,$ then there is a *semi*α-open* subset A and an open set B such that $x\in A\subseteq U$ and $x\in B\subseteq V.$ Then $x\in A\cap B\subseteq U\cap V.$ Since the intersection of an open set and *semi*α-open* set is *semi*α-open.* Thus $A\cap B\in s^{*}\alpha\text{-}O(X).$ Therefore $U\cap V$ is a *semi*α-neighborhood* of $x.$

(5) Let U be a *semi*α-neighborhood* for zero. Then there exists a *semi*α-open* neighborhood V of zero such that $V\subseteq U.$ Since the map $M_{\lambda}:X\to X,$ defined by $M_{\lambda}(x)=\lambda x,$ is *semi*α-irresolute.* The inverse map $N_{\lambda};X\to X,$ defined by $N_{\lambda}(x)=\frac{1}{\lambda}x,$ is also a *semi*α-irresolute.* Thus M_{λ} is *semi*α-homeomorphism,* for each $\lambda\in R-\{0\}.$ Hence $M_{\lambda}(V)=\lambda V$ is a *semi*α-open* neighborhood of zero. Furthermore, clearly $\lambda V\subseteq\lambda U.$ Thus consequently $\lambda U\in s^{*}\alpha\text{-}N(0).$

(7) Suppose U is a *semi*α-neighborhood* for zero. Then there exists a *semi*α-open* neighborhood V of zero such that $V\subseteq U.$ Since the map $T_{x}:X\to X,$ defined by $T_{x}(y)=y+x,$ is *semi*α-irresolute.* The inverse map $S_{x};X\to X,$ defined by $S_{x}(y)=x-y,$ is also *semi*α-irresolute.* Thus T_{x} is *semi*α-homeomorphism,* for each $x\in X.$ Hence $T_{x}(V)=x+V$ is a *semi*α-open* neighborhood for a point $x.$ Clearly $x+V\subseteq x+U.$ Thus $x+U\in s^{*}\alpha\text{-}N(x).$ The converse can be proved similarly.

Definition 4.2. A subset A of a topological vector space X is called balanced if and only if $\lambda A\subseteq A$ for each $\lambda\in R$ such that $|\lambda|\leq 1.$

Definition 4.3. A subset A of a topological vector space X is called absorbing if for all $x \in X$ there exists a number $\varepsilon > 0$ such that $\lambda x \in A$ for $|\lambda| \leq \varepsilon$.

Definition 4.4. A set C of a topological vector space X is said to be convex, if and only if it contains all segments between its points: $x \in C$, $y \in C$, for $t \in [0,1]$ implies $tx+(1-t)y \in C$, or equivalently $tC+(1-t)C \subseteq C$, for all $t \in [0,1]$. A set C of a topological vector space X is said to be absolutely convex if it is both convex and balanced.

Theorem 4.5. Let $(X_{(K)}, \tau)$ be a *semi*α-irresolute* topological vector space. If a subset C of X is convex, then $s^*\alpha\text{-}Cl(C)$ is also convex.

Proof. The convexity of C implies $tC+(1-t)C \subseteq C$. By Theorem 3.11 (2), and Theorem 3.12, it follows that $t[s^*\alpha\text{-}Cl(C)]+(1-t)[s^*\alpha\text{-}Cl(C)] = s^*\alpha\text{-}Cl(tC)+ s^*\alpha\text{-}Cl[(1-t)C] \subseteq s^*\alpha\text{-}Cl[tC+(1-t)C] = s^*\alpha\text{-}Cl(C)$. Thus $t[s^*\alpha\text{-}Cl(C)]+(1-t)[s^*\alpha\text{-}Cl(C)] \subseteq s^*\alpha\text{-}Cl(C)$. Hence we conclude that $s^*\alpha\text{-}Cl(C)$ is convex.

Theorem 4.6. Let $(X_{(K)}, \tau)$ be a *semi*α-irresolute* topological vector space. If a subset C of X is convex, then $s^*\alpha\text{-}Int(C)$ is also convex.

Proof. By Theorem 3.5, it follows immediately that $s^*\alpha\text{-}Int(A)+s^*\alpha\text{-}Int(B) \subseteq s^*\alpha\text{-}Int(A+B)$ for any subsets A and B of X. Then applying this result and Theorem 3.7 (2), $t[s^*\alpha\text{-}Int(C)]+(1-t)[s^*\alpha\text{-}Int(C)] = s^*\alpha\text{-}Int(tC)+ s^*\alpha\text{-}Int[(1-t)C] \subseteq s^*\alpha\text{-}Int[tC+(1-t)C] = s^*\alpha\text{-}Int(C)$. Therefore $s^*\alpha\text{-}Int(C)$ is convex.

Theorem 4.7. Let $(X_{(K)}, \tau)$ be a *semi*α-irresolute* topological vector space. Then the following statements are equivalent:

(a) Every *semi*α-neighborhood* U of zero is absorbing.

(b) For every *semi*α-neighborhood* U for zero, there exists a balanced set $V \in s^*\alpha\text{-}N(0)$ such that $V \subseteq U$.

Proof. (a) Suppose U is a *semi*α-neighborhood* for zero. Then there exists a *semi*α-open* subset $V \in N_0$ such that $V \subseteq U$. By hypothesis X is a *semi*α-irresolute* topological vector space. So there exist *semi*α-open* sets V_1 of containing *zero* and V_2 of X containing zero such that $V_1.V_2 \subseteq V$. The set V_1 contains an open interval of the form $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Therefore $tx \in V \subseteq U$ for all $t \in (-\varepsilon, \varepsilon)$ and for all $x \in V_2$. This implies U is absorbing.

(b) Let U be a *semi*α-neighborhood* of zero. By hypothesis X is a *semi*α-irresolute* topological vector space. So there exist *semi*α-open* sets V_1 of containing zero and V_2 of X containing zero such that $V_1.V_2 \subseteq U$. Then there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq V_1$. Define $W = \cup\{tV_2 : t \in R, |t| < \varepsilon\}$. Since tV_2 is a *semi*α-neighborhood* of zero, for $t \neq 0$ and $tV_2 \subseteq U$ for $t \in (-\varepsilon, \varepsilon)$. Thus W is a *semi*α-neighborhood* for zero and $W \subseteq U$. Now we have to show that W is balanced. Let $r \in R$ such that $|r| \leq 1$. Let $t \in (-\varepsilon, \varepsilon)$ and $x \in V_2$. Since $|rt| = |r||t| \leq |t| < \varepsilon$. Thus $r(tx) = (rt)x \in (-\varepsilon, \varepsilon).V_2 \subseteq W$. This shows that $rW \subseteq W$. Therefore W is balanced.

Theorem 4.8. Let X be a *semi*α-irresolute* topological vector space. Then $s^*\alpha\text{-}Cl(A) = \cap\{A+U : U \in s^*\alpha\text{-}N(0)\}$.

Proof. Assume $x \in s^*\alpha\text{-}Cl(A)$ and let U be a *semi*α-neighborhood* of zero. Then by Theorem 4.7(b), there exits a balanced neighborhood V for zero such that $V \subseteq U$. Thus $x+V$ is a *semi*α-neighborhood* for x and $x \in s^*\alpha\text{-}Cl(A)$,

so $(x+V)\cap A\neq\phi$. Take $a\in(x+V)\cap A$. Then $a\in x+V$ and $a\in A$. Let $a=x+v$ for some $v\in V$. Since V is balanced, so $A-V=A+V$. Take $x=a+(-v)\in A-V$ implies $x\in A+V\subseteq A+U$. Thus $x\in A+U$, for any *semi*α-neighborhood* U of zero. Thus we obtain $s^*\alpha\text{-}Cl(A)\subseteq\cap\{A+U:U\in s^*\alpha\text{-}N(0)\}$.

Conversely if $x\notin s^*\alpha\text{-}Cl(A)$, then there exists a balanced neighborhood U for zero such that $(x+U)\cap A=\phi$. Thus $x\notin A-U=A+U$. Hence $x\notin\cap\{A+U:U\in s^*\alpha\text{-}N(0)\}$. It follows that $\cap\{A+U:U\in s^*\alpha\text{-}N(0)\}\subseteq s^*\alpha\text{-}Cl(A)$. Thus we get $s^*\alpha\text{-}Cl(A)=\cap\{A+U:U\in s^*\alpha\text{-}N(0)\}$.

Theorem 4.9. Let X be a *semi*α-irresolute* topological vector space. Then the following assertions are true.

(a) For every $U\in s^*\alpha\text{-}N(0)$, there exists symmetric set $V\in s^*\alpha\text{-}N(0)$ such that $V+V\subseteq U$.

(b) For every $U\in s^*\alpha\text{-}N(0)$, there exits a *semi*α-closed* balanced set $V\in s^*\alpha\text{-}N(0)$ such that $V\subseteq U$.

(c) For every $U\in s^*\alpha\text{-}N(0)$, there exists symmetric set $V\in s^*\alpha\text{-}N(0)$ such that $V+V+V\subseteq U$.

Proof. (a) Assume $U\in s^*\alpha\text{-}N(0)$. By hypothesis X is a *semi*α-irresolute* topological vector space. There exist *s*α-open* neighborhoods V_1 and V_2 for zero in X such that $V_1+V_2\subseteq U$. Let $V=V_1\cap(-V_1)\cap V_2\cap(-V_2)$. Then V is a symmetric *semi*α-open* neighborhood of zero and $V+V\subseteq V_1+V_2\subseteq U$.

(b) Let U be a *semi*α-neihborhood* of zero in X. By part (a), there is *semi*α-neihborhood* V for zero with $V+V\subseteq U$. By Theorem 4.7 (b), there exits *semi*α-neihborhood* W for zero, which is balanced and $W\subseteq V$. Then by Theorem 4.8, we obtain $s^*\alpha\text{-}Cl(W)\subseteq W+V\subseteq V+V\subseteq U$. This shows that U contains a *semi*α-closed* neighborhood of zero.

(c) Follows easily from (a).

Definition 4.10. A topological space (X,τ) is called *semi*α-Hausdorff*, if for each two distinct points x and y in X, there exist disjoint *semi*α-open* sets U, V such that $x\in U$ and $y\in V$.

Now we give some properties of *semi*α-Hausdorff* space.

Theorem 4.11. Let X be a *semi*α-irresolute* topological vector space. Then the following statements are equivalent.

(a) X is *semi*α-Hausdorff*.

(b) If $x\in X$, $x\neq 0$, then there exists $U\in s^*\alpha\text{-}N(0)$ such that $x\notin U$.

(c) If $x,y\in X$, $x\neq y$, then there exists $V\in s^*\alpha\text{-}N(x)$ such that $y\notin V$.

Proof. By continuity of translation, it is sufficient to prove the equivalence between (a) and (b) only.

(a)$\Rightarrow$(b): Assume a non-zero vector x belongs to X. Since X is *semi*α-Hausdorff*. Therefore there are disjoint *semi*α-open* sets $U,V\subseteq X$ such that $0\in U$ and $x\in V$. Thus $U\in s^*\alpha\text{-}N(0)$, $V\in s^*\alpha\text{-}N(x)$ and $x\notin U$.

(b)$\Rightarrow$(a): Let $x,y\in X$ be such that $x-y\neq 0$. Then there exists $U\in s^*\alpha\text{-}N(0)$ such that $x-y\notin U$. By Theorem 4.9

(a), there exists a symmetric *semi*α-neihborhood* W of zero such that $W+W\subseteq U$. By Theorem 4.7 (b), W can be assumed to be balanced. Let $V_1=x+W$ and $V_2=y+W$. We note that $V_1\in s^*\alpha\text{-}N(x)$, $V_2\in s^*\alpha\text{-}N(y)$ and $V_1\cap V_2=\phi$, since if $z\in V_1\cap V_2$, then $z-x\in W$ and $z-y\in W$. Since W is balanced, so $-(z-x)\in W$. It follows that $x-y=(z-y)+[-(z-x)]\in W+W\subseteq U$, which is a contradiction. So, we must have $V_1\cap V_2=\phi$. Finally, by the definition of *semi*α-neihborhood*, there exist $V_1^*, V_2^*\in s^*\alpha\text{-}O(X)$ such that $x\in V_1^*\subseteq V_1$, $y\in V_2^*\subseteq V_2$, and $V_1^*\cap V_2^*=\phi$. This shows that the space X is *semi*α-Hausdorff*. This completes the proof.

The following result follows from Theorem 4.11.

Corollary 4.12. Let X be a *semi*α-irresolute* topological vector space. Then the following statements are equivalent.

(a) X is *semi*α-Hausdorff*.

(b) $\cap\{U: U\in s^*\alpha\text{-}N(0)\}=\{0\}$.

(c) $\cap\{U: U\in s^*\alpha\text{-}N(x)\}=\{x\}$.

Theorem 4.13. A *semi*α-irresolute* topological vector space X is *semi*α-Hausdorff* if and only if every one-point set in X is *semi*α-closed* in X.

Proof. Let $x\in X$ and $y\in X-\{x\}$. Then $y-x\neq 0$, and by assumption, there exists $U\in s^*\alpha\text{-}N(0)$ such that $y-x\notin U$. By Theorem 4.9 (b), there exists a *semi*α-closed* and balanced set $V\in s^*\alpha\text{-}N(0)$ such that $V\subseteq U$. It follows that $y-x\notin V$ that is $y-x\in X-V$. Thus $y\in(X-V)+\{x\}$. But $(X-V)+\{x\}$ is *semi*α-open*, since V is *semi*α-closed*, and $(X-V)+\{x\}\subseteq X-\{x\}$. This shows that $X-\{x\}$ is *semi*α-open*. For the converse, let $x\in X$ and assume that $\{x\}$ is *semi*α-closed*. Then by Theorem 4.8, $\{x\}=s^*\alpha\text{-}Cl(\{x\})=\cap\{U+\{x\}: U\in s^*\alpha\text{-}N(0)\}=\cap\{V: V\in s^*\alpha\text{-}N(x)\}$, where $V=U+\{x\}\in s^*\alpha\text{-}N(x)$. Then by Corollary 4.12, X is *semi*α-Hausdorff*. This completes the proof.

Since translation is a *semi*α-homeomorphism* and as a consequence of Theorem 4.13, we have the following result.

Corollary 4.14. A *semi*α-irresolute* topological vector space X is *semi*α-Hausdorff* if and only if $\{0\}$ is *semi*α-closed* in X.

Theorem 4.15. Let C, M be disjoint sets in a *semi*α-irresolute* topological vector space X with C *semi*α-closed*, M *semi*α-compact*. Then there exists $U\in s^*\alpha\text{-}N(0)$ with $(M+U)\cap(C+U)=\phi$.

Proof. If $M=\phi$, then there is nothing to prove. Otherwise, let $x\in M$ and by the invariance with translation, we can assume $x=0$. Then $X-C$ is a *semi*α-open* neighborhood of zero. Since addition is *semi*α-irresolute* and *semi*α-continuous*, by $0=0+0+0$, there is an open neighborhood U of 0 such that $3U=U+U+U\subseteq X-C$. By defining $W=U\cap(-U)\subseteq U$ we have that W is open, and hence *semi*α-open* symmetric and $3W=W+W+W\subseteq X-C$. This means that $\phi=\{3x: x\in W\}\cap C=\{2x: x\in W\}\cap\{y-x: y\in C, x\in W\}\supseteq W\cap(C+W)$. This concludes the proof for a single point.

Sine M is *semi*α-compact*, then repeating the above argument for all $x\in M$, we obtain symmetric *semi*α-open* sets V_x such that $(x+2V_x)\cap(C+V_x)=\phi$. The set $\{x+V_x: x\in M\}$ is a *semi*α-open* covering of M, but M is *semi*α-compact*. Hence there is a finite number of points $x_i\in M$, $i=1,2,...,n$, such that $M\subseteq\bigcup_{i=1}^{n}(x_i+V_{x_i})$. Define the

*semi*α-open* neighborhood V of zero by $V=\bigcap_{i=1}^{n}V_{x_i}$. Then we get $(M+V)\cap(C+V)\subseteq \bigcup_{i=1}^{n}(x_i+V_{x_i}+V)\cap(C+V)\subseteq \bigcup_{i=1}^{n}\left[(x_i+2V_{x_i})(C+V_{x_i})\right]=\phi$. This completes the proof.

Lemma 4.16. If U is a *semi*α-open* set and $U\cap A=\phi$, then $U\cap\left[s^*\alpha Cl(A)\right]=\phi$.

Proof. Suppose that there exists an $x\in U\cap\left[s^*\alpha\text{-}Cl(A)\right]$. Then $x\in s^*\alpha\text{-}Cl(A)$ and U is a *semi*α-open* neighborhood of x and $X-U$ is *semi*α-closed* set containing A, hence $s^*\alpha\text{-}Cl(A)\subseteq X-U$ and $x\notin s^*\alpha\text{-}Cl(A)$ which is contradiction, hence $U\cap\left[s^*\alpha\text{-}Cl(A)\right]=\phi$.

Corollary 4.17. Let C, M be disjoint sets in a *semi*α-irresolute* topological vector space X with C *semi*α-closed*, M *semi*α-compact*. Then there exists $U\in s^*\alpha\text{-}N(0)$ with $\left[s^*\alpha\text{-}Cl(M+U)\right]\cap(C+U)=\phi$.

Proof. By Theorem 4.15, there exists $U\in s^*\alpha\text{-}N(0)$ such that $(M+U)\cap(C+U)=\phi$. Now $C+U=\bigcup\{y+U:y\in C\}$ is *semi*α-open* set being a union of *semi*α-open* sets. Then by Lemma 4.16, we obtain $\left[s^*\alpha\text{-}Cl(M+U)\right]\cap(C+U)=\phi$.

Theorem 4.18. Let $\left(X_{(K)},\tau\right)$ be a *semi*α-irresolute* topological vector space. Let $f:X\to$ □ be a *non-zero* linear map. Then $f(G)$ is *semi*α-open* in □ whenever G is *semi*α-open* in X.

Proof. Let G be a nonempty *semi*α-open* set. Then one can assume that there is $x_0\in X-\{0\}$ such that $f(x_0)=1$. For any $a\in G$, it is required to show that $f(a)\in s^*\alpha\text{-}Int\left[f(G)\right]$. Since $G\in s^*\alpha\text{-}N(a)$ by Theorem 4.1 we have $G-a\in s^*\alpha\text{-}N(0)$. Also by Theorem 4.7 (a) $G-a$ is absorbing, that is, absorbs x_0, namely there exists an $\varepsilon>0$ such that $\lambda x_0\in G-a$ whenever $\lambda\in$ □ with $|\lambda|\le\varepsilon$. Now for any $\beta\in$ □ with $|\beta-f(a)|\le\varepsilon$ we have $\left[\beta-f(a)\right]x_0\in G-a$, so $f\left[(\beta-f(a))x_0\right]\in f(G-a)$. Since f is linear. This implies that $(\beta-f(a))f(x_0)\in f(G-a)$. Hence we get $(\beta-f(a))(1)\in f(G-a)=f(G)-f(a)$. This implies that $\beta\in f(G)$ and $f(a)\in(\beta-\varepsilon,\beta+\varepsilon)$. Thus $f(a)\in Int\left[f(G)\right]\subseteq s^*\alpha\text{-}Int\left[f(G)\right]$; hence consequently $f(G)=s^*\alpha\text{-}Int\left[f(G)\right]$.

Lemma 4.19. Let X be a vector space and $\phi\neq M\subseteq X$. For $a\in M$, the following statements are equivalent.

(1) a is an extreme point of M.

(2) If $x, y\in M$ are such that $a=\frac{1}{2}(x+y)$, then $a=x=y$.

(3) Let $x, y\in M$ be such that $x\neq y$, let $\lambda\in(0,1)$ and $a=\lambda x+(1-\lambda)y$. Then we have either $\lambda=0$ or $\lambda=1$.

Theorem 4.20. Let $\left(X_{(K)},\tau\right)$ be a *semi*α-irresolute* topological vector space and C be a convex subset of X. Then $\left[s^*\alpha\text{-}Int(C)\right]\cap(\partial C)=\phi$.

Proof. If $s^*\alpha\text{-}Int(C)=\phi$, then the result is trivial. Suppose that $s^*\alpha\text{-}Int(C)\neq\phi$ and let $x\in s^*\alpha\text{-}Int(C)$. Then there exists $V\in s^*\alpha\text{-}N(0)$ such that $x+V\subseteq C$. As the map $\Phi:$ □ $\to X$, where $\Phi(\mu)=\mu x$ is continuous at $\mu=1$, for this *semi*α-neighborhood* $x+V$, there is an $r>0$ such that $\mu x\in x+V$ whenever $|\mu-1|\le r$. In particular, we have $(1+r)x\in x+V\subseteq C$ and $(1-r)x\in x+V\subseteq C$. Now consider $x=\lambda(1+r)x+(1-\lambda)(1-r)x$ and take $\lambda=\frac{1}{2}$. Consequently,

we have $x=\frac{1}{2}(1+r)x+\left(1-\frac{1}{2}\right)(1-r)x,$ which implies that x is not an extreme point of $C.$

Theorem 4.21. Let X be a *semi*α-irresolute* topological vector space and W an *s*α-neighborhood* of 0. Then there is a *semi*α-neighborhood* U of 0 such that $s^*\alpha\text{-}Cl(U)\subseteq W.$ Equivalently, if C is a *semi*α-closed* subset of X and x a point of X outside C then there are disjoint *semi*α-open* sets U_1 and U_2 with $x\in U_1$ and $C\subseteq U_2.$

Proof. Let x be a point outside a *semi*α-closed* set $C\subseteq X.$ We will produce a *semi*α-open* set U containing x with $\left[s^*\alpha\text{-}Cl(U)\right]\cap C=\phi;$ then $U_1=U$ and $U_2=X-\left[s^*\alpha\text{-}Cl(U)\right],$ the complement of the *semi*α-closure* of $U,$ are disjoint *semi*α-open* sets with $x\in U_1$ and $C\subseteq U_2,$ as desired. We know that X looks the same everywhere, so we may work with $x=0.$ Let W be the complement of $C.$ Then W is a *semi*α-open* set with $0\in W.$ By hypothesis X is *semi*α-irresolute* topological vector space. So by Theorem 4.9 (a), there exists a *semi*α-open* subset U of 0 such that $U+U\subseteq W.$ This means that $U+U$ is disjoint from $C.$ Equivalently, U is disjoint from $C-U.$ For otherwise there would be an $x\in U$ which could be expressed as $c-y$ with $c\in C$ and $y\in U,$ which would imply that $c=x+y\in U+U\subseteq W$ is in $W.$ Now the set $-U$ is *semi*α-open* because the map $X\to X:\ x\to(-1)x=-x$ is a *semi*α-homeomorphism,* and hence so are all its translates $x-U.$ So the set $U_2=C-U=\bigcup\{c-U:c\in C\}$ is *semi*α-open,* being the union of *semi*α-open* sets. Thus we have found a *semi*α-open* set $U_1=U$ containing 0 and a *semi*α-open* set $U_2,$ disjoint from $U_1,$ with $C\subseteq U_2.$

Theorem 4.22. Let $\left(X_{(K)},\tau\right)$ be a *semi*α-irresolute* topological vector space. Then every *semi*α-open* subspace S of X is also a *semi*α-irresolute* topological vector space.

Proof. Let W be a *semi*α-open* neighborhood of $x+y$ in S where x, y are two distinct points in $S.$ Since S is a *semi*α-open* subspace of $X,$ then W is a *semi*α-open* neighborhood of $x+y$ in $X,$ and by definition of *semi*α-irresolute* topological vector space, there exist *semi*α-open* neighborhoods U of x in X and V of y in X such that $U+V\subseteq W.$ Then the sets $G=U\cap S$ and $H=V\cap S$ are *semi*α-open* neighborhoods of x and y respectively, in S such that $G+H\subseteq U+V\subseteq W.$ Now suppose $\lambda\in K,\ x\in S$ and let W be a *semi*α-open* neighborhood of λx in $S.$ Since S is a *semi*α-open* subspace of $X,$ then W is a *semi*α-open* neighborhood of λx in $X.$ Then there exist *semi*α-open* neighborhoods U of λ in K and V of x in X such that $U.V\subseteq W.$ Then the set $G=U$ is a *semi*α-open* neighborhood of λ in K and the set $H=V\cap S$ is a *semi*α-open* neighborhood of x in $S.$ Also $G.H\subseteq U.V\subseteq W.$ Hence S is a *semi*α-irresolute* topological vector space.

Theorem 4.23. Suppose that $\left(X_{(K)},\tau\right)$ is a *semi*α-irresolute* topological vector space. If $S\subseteq X$ is a linear subspace, then so is $s^*\alpha\text{-}Cl(S).$

Proof. Let S be a linear subspace of $X.$ Thus $S+S\subseteq S$ and for all $\lambda\in K, \lambda.S\subseteq S.$ By Theorem 3.12, $s^*\alpha\text{-}Cl(S)+s^*\alpha\text{-}Cl(S)\subseteq s^*\alpha\text{-}Cl(S+S)\subseteq s^*\alpha\text{-}Cl(S).$ By Theorem 3.11(2), for every $\lambda\in K,$ $\lambda\left[s^*\alpha\text{-}Cl(S)\right]=s^*\alpha\text{-}Cl(\lambda S)\subseteq s^*\alpha\text{-}Cl(S).$ Therefore, $s^*\alpha\text{-}Cl(S)$ is a linear subspace of $X.$

Definition 4.24. Suppose that $\left(X_{(K)},\tau\right)$ is a *semi*α-irresolute* topological vector space. A subset $E\subseteq X$ is said to be bounded if for all *semi*α-open* sets V containing 0, there exists $s\in R$ such that for all $t>s,\ E\subseteq tV.$ That is, every *semi*α-open* neighborhood of zero contains E after being blown up sufficiently.

Theorem 4.25. Suppose that $\left(X_{(K)},\tau\right)$ is a *semi*α-irresolute* topological vector space. If E is a bounded subset of $X,$ then $s^*\alpha\text{-}Cl(E)$ is bounded.

Proof. Let W be an $s^*\alpha$-*open* set containing 0, then by Theorem 4.21, there exists $U \in s^*\alpha\text{-}N(0)$ such that $s^*\alpha\text{-}Cl(U) \subseteq W$. Since E is bounded, so $E \subseteq tU \subseteq t\left[s^*\alpha\text{-}Cl(U)\right] \subseteq tW$, for sufficiently large values of t. It follows that for large enough t, $s^*\alpha\text{-}Cl(E) \subseteq s^*\alpha\text{-}Cl(tU) \subseteq t\left[s^*\alpha\text{-}Cl(U)\right] \subseteq tW$. Thus, $s^*\alpha\text{-}Cl(E)$ is bounded.

Theorem 4.26. Let (X,τ) be a *semi*$^*\alpha$-*irresolute* topological vector space. Let V be a *semi*$^*\alpha$-*open* neighborhood of zero in X. Then for every sequence $\{r_n : n \in N\}$ of positive real numbers such that $r_n \to \infty$, $\bigcup_{n=1}^{\infty} r_n V = X$.

Proof. Let $x \in X$ and consider the sequence $\left\{\frac{x}{r^n} : n \in N\right\}$. This sequence converges to 0 by the *semi*$^*\alpha$-*irresoluteness* of the scalar multiplication $F \times X \to X$. Thus, for sufficiently large n, $\frac{x}{r^n} \in V$ i.e., $x \in r_n V$.

Theorem 4.27. Let (X,τ) be a *semi*$^*\alpha$-*irresolute* topological vector space. Then every *semi*$^*\alpha$-*compact* set is bounded.

Proof. Let C be a *semi*$^*\alpha$-*compact* subset of X. We need to prove that it is bounded, namely, that for every *semi*$^*\alpha$-*open* neighborhood V of 0, $C \subseteq tV$ for sufficiently large t. Let V be a *semi*$^*\alpha$-*open* neighborhood of 0, then by Theorem 4.7(b), there exists a balanced *semi*$^*\alpha$-*open* neighborhood W of 0 such that $W \subseteq V$. By Theorem 4.26, $C \subseteq \bigcup_{n=1}^{\infty} nW$. Since, C is *semi*$^*\alpha$-*compact*, therefore there exists a positive integer m such that $C \subseteq \bigcup_{j=1}^{m} n_j W = n_m \bigcup_{j=1}^{m} (n_j / n_m) W \subseteq n_m W$. Thus, for all $t > n$, $C \subseteq n_m W = \left[t(n_m / t)\right] W \subseteq tW \subseteq tV$, which implies that C is bounded.

Theorem 4.28. Let (X,τ) be a *semi*$^*\alpha$-*irresolute* topological vector space. Then every Cauchy sequence in X is bounded.

Proof. Let $\{x_n : n \in N\}$ be a Cauchy sequence in X. Let W be a *semi*$^*\alpha$-*open* neighborhood of zero, then by Theorem 4.9 (a), there exists a *semi*$^*\alpha$-*open* neighborhood V of 0 such that $V + V \subseteq W$. By definition of a Cauchy sequence, there exists N such that for all $m, n \geq N$, $x_n - x_m \in V$ and in particular for all $n > N$, $x_n \in x_N + V$. Set $s > 1$ such that $x_N \in sV$, then for all $n > N$, $x_n \in sV + V \subseteq sV + sV \subseteq sW$. Since for balanced sets $sW \subseteq tW$ for $s < t$, and since every *semi*$^*\alpha$-*open* neighborhood of 0 contains a balanced neighborhood, this proves that the sequence is bounded.

Definition 4.29. Let X be a vector space over $\square$. A non-negative real-valued function p defined on X is a *pseudonorm* if it satisfies the following two conditions.

(i) $p(\lambda x) = |\lambda| p(x)$, for all $x \in X$ and $\lambda \in \square$;

(ii) $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$.

Now, we introduce the notion of locally convex *semi*$^*\alpha$-*irresolute* topological vector space. Moreover, we give a necessary and sufficient condition, in terms of convex *semi*$^*\alpha$-*neighborhoods* of 0, for a *semi*$^*\alpha$-*irresolute* topological vector space to be locally convex.

Definition 4.30. A *semi*$^*\alpha$-*irresolute* topological vector space (X,τ) is locally convex if for all $x \in X$, every $S \in s^*\alpha\text{-}N(x)$ contains a convex set $U \in s^*\alpha\text{-}N(x)$.

Theorem 4.31. A *semi*$^*\alpha$-*irresolute* topological vector space (X,τ) is locally convex if and only if every

$S \in s^*\alpha\text{-}N(0)$ contains a convex set $U \in s^*\alpha\text{-}N(0)$.

Proof. The sufficiency part is trivial. Let $S \in s^*\alpha\text{-}N(x)$. Then by Theorem 4.1(7), $S - x \in s^*\alpha\text{-}N(0)$ and by assumption, there exists a convex set $U \in s^*\alpha\text{-}N(0)$ such that $U \subseteq S - x$. Hence by Theorem 4.1 (7) again, $U + x \in s^*\alpha\text{-}N(x)$. As $U + x \subseteq S$ and as $U + x$ is convex, (X, τ) is a locally convex *semi*α-irresolute* topological vector space.

Corollary 4.32. In a locally convex *semi*α-irresolute* topological vector space (X, τ), a *pseudonorm* p is *semi*α-irresolute* if and only if p is *semi*α-irresolute* at zero.

Proof. If p is *semi*α-irresolute,* then p is *semi*α-irresolute* at zero. Conversely, suppose p is *semi*α-irresolute* at 0, and let $x \in X$ and $V \in N_{p(x)}(\square)$. Then by Theorem 4.1 (7), $V - p(x) \in N_0(\square) = N_{p(0)}(\square)$ and thus $(-\varepsilon, \varepsilon) \subseteq V - p(x)$ for some $\varepsilon > 0$. Clearly $(-\varepsilon, \varepsilon)$ being an open set in $\square$ is *semi*α-open* set in $\square$. By assumption, there exists $U \in s^*\alpha\text{-}N(0)$ such that $p(U) \subseteq (-\varepsilon, \varepsilon)$ and as $p(y) \geq 0$ for all $y \in U$, $p(U) \subseteq [0, \varepsilon)$. Then by Theorem 4.1 (7), $U + x \in s^*\alpha\text{-}N(x)$. For all $y \in U$, $0 \leq p(x+y) \leq p(x) + p(y) \leq p(x) + \varepsilon$, $p(x+y) \in [0, p(x) + \varepsilon)$. Therefore it follows that $p(U + x) \subseteq V$.

Definition 4.33. Let A be an absolutely convex subset of a vector space X. Then the functional defined by $p(x) = inf\{\lambda : \lambda > 0, x \in \lambda A\}$ is called the *gauge* of A.

Lemma 4.34. In a vector space X, the *gauge* of an absolutely convex and absorbent subset is a *pseudonorm.*

Now, we prove the main result in which we characterize absolutely convex and absorbent *semi*α-neighborhoods* of zero in terms of their *semi*α-irresolute gauges*.

Theorem 4.35. Let p be a *gauge* of an absolutely convex and absorbent subset U of a *semi*α-irresolute* topological vector space (X, τ). Then p is *semi*α-irresolute* if and only if U is a *semi*α-neighborhood* of zero.

Proof. If p is *semi*α-irresolute,* then as $(-1, 1)$ is a *semi*α-open* set in $\square$, $V = \{x : p(x) < 1\} = p^{-1}[(-1, 1)]$ is a *semi*α-open* subset of X. Thus as $V \subseteq U$, $U \in s^*\alpha\text{-}N(0)$. Conversely, if $U \in s^*\alpha\text{-}N(0)$ and $\varepsilon > 0$, then by Theorem 4.1 (5), $V = \varepsilon U \in s^*\alpha\text{-}N(0)$ and $p(x) < \varepsilon$ for all $x \in V$. Thus $p(V) \subseteq (-\varepsilon, \varepsilon)$. Hence, p is *semi*α-irresolute* at zero. By Lemma 4.34, p is a *pseudonorm* and by Corollary 4.32, p is *semi*α-irresolute* at each $x \in X$. Therefore p is *semi*α-irresolute.*

5. CONTRA SEMI*α-TOPOLOGICAL VECTOR SPACES

In this section we present contra semi*α-continuous functions and investigate some of their properties and characterizations. Then we introduce a new form of topological vector space called contra semi*α-topological vector space and investigate its several properties and characterizations.

Definition 5.1. A function $f : (X, \tau) \to (Y, \sigma)$ is called a contra continuous function if $f^{-1}(V)$ is open set in (X, τ) for every closed set V of Y.

Equivalently A function $f : (X, \tau) \to (Y, \sigma)$ is called a contra continuous function if $f^{-1}(V)$ is closed set in (X, τ) for every open set V of Y.

Definition 5.2. A function $f:(X,\tau)\to(Y,\sigma)$ is called a contra semi$^*\alpha$-continuous function if $f^{-1}(V)$ is semi$^*\alpha$-open set in (X,τ) for every closed set V of (Y,σ).

Theorem 5.3. If a function $f:(X,\tau)\to(Y,\sigma)$ is a contra continuous function, then f is a contra semi$^*\alpha$-continuous function.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be a contra continuous function. Let V be an open set in (Y,σ). Since f is contra continuous, then $f^{-1}(V)$ is closed in (X,τ). Since $C(X,\tau)\subseteq semi^*\alpha C(X,\tau)$. Hence $f^{-1}(V)$ is semi$^*\alpha$-closed in (X,τ). Therefore, f is contra semi$^*\alpha$-continuous.

Example 5.4. A contra semi$^*\alpha$-continuous function does not imply contra continuous function. Let X be a space of real numbers with the usual topology τ induced on it. Consider $M=\left\{\frac{1}{n}:n\in\square\right\}$, where $\square$ being the set of natural numbers. Then M^c is a semi$^*\alpha$-open set which is not open. Let $Y=\{1,2\}$ with topology $\sigma=\{\phi,\{1\},Y\}$. Define $f:(X,\tau)\to(Y,\sigma)$ by $f(x)=1$ for each $x\in M$ and $f(x)=2$ for each $x\in X-M$. Then $f^{-1}(\{1\})=M$ and M is a semi$^*\alpha$-closed set which is not closed. Therefore, f is a contra semi$^*\alpha$-continuous function and not a contra continuous function.

Definition 5.5. Let (X,τ) be a topological space and A be a subset of X. A point $x\in X$ is said to be kernelled point of A $(Briefly,\ x\in Ker(A))$ if and only if for each closed set B contains x, then $B\cap A\neq\phi$.

Lemma 5.6. Let (X,τ) be a topological space and $A,\ B$ be subsets of X. Then the following properties hold:

(i) $Ker(A)=\cap\{U\in\tau:A\subseteq U\}$.

(ii) $A\subseteq Ker(A)$ and $A=Ker(A)$ if A is open in X.

(iii) If $A\subseteq B$, then $Ker(A)\subseteq Ker(B)$.

Definition 5.7. Let N be a subset of a topological space (X,τ) and let $x\in X$. Then N is called a semi$^*\alpha$-neighborhood of x, if there exists a semi$^*\alpha$-open set U containing x such that $U\subseteq N$.

Theorem 5.8. Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then the following statements are equivalent:

(1) f is contra semi$^*\alpha$-continuous.

(2) For each $x\in X$ and each closed set V of Y with $f(x)\in V$, there exists an semi$^*\alpha$-open set U of X containing x such that $f(U)\subseteq V$.

(3) For each $x\in X$ and each closed set V of Y with $f(x)\in V$, $f^{-1}(V)$ is an semi$^*\alpha$-neighborhood of x.

(4) $f^{-1}(V)$ is semi$^*\alpha$-closed in X, for every open set V in Y.

(5) $f\left[s^*\alpha\text{-}Cl(A)\right]\subseteq Ker\left[f(A)\right]$, for every subset A of X.

(6) $s^*\alpha\text{-}Cl\left[f^{-1}(B)\right]\subseteq f^{-1}\left[Ker(B)\right]$, for every subset B of Y.

Proof. $(1)\Rightarrow(2)$: Let $x\in X$ and V be a closed set in Y such that $f(x)\in V$. Since f is contra semi$^*\alpha$-continuous, $f^{-1}(V)\in s^*\alpha$-$O(X,\tau)$. By putting $U=f^{-1}(V)$ which is containing x, we have $f(U)\subseteq V$.

$(2)\Rightarrow(3)$: Let V be a closed set in Y and $f(x)\in V$. Then by (2), there exists an semi$^*\alpha$-open set U of X containing x such that $f(U)\subseteq V$. So $x\in U\subseteq f^{-1}(V)$. Hence $f^{-1}(V)$ is an semi$^*\alpha$-neighborhood of x.

$(3)\Rightarrow(1)\Rightarrow(4)$: Easy.

$(1)\Rightarrow(5)$: Let A be any subset of X. Suppose that $y\notin Ker[f(A)]$. Then there exists a closed set V of Y containing y such that $f(A)\cap V=\phi$. Thus, we have $A\cap f^{-1}(V)=\phi$ and $s^*\alpha$-$Cl(A)\cap f^{-1}(V)=\phi$. Therefore, we obtain $f[s^*\alpha$-$Cl(A)]\cap V=\phi$ and $y\notin f[s^*\alpha$-$Cl(A)]$. This implies that $f[s^*\alpha$-$Cl(A)]\subseteq Ker[f(A)]$.

$(5)\Rightarrow(6)$: Let B be any subset of Y. By (5) and Lemma 5.6, we have $f[s^*\alpha$-$Cl(f^{-1}(B))]\subseteq Ker[f(f^{-1}(B))]\subseteq Ker(B)$ and $s^*\alpha$-$Cl[f^{-1}(B)]\subseteq f^{-1}[Ker(B)]$.

$(6)\Rightarrow(4)$: Let V be any open set of Y. Then by Lemma 5.6, we have $s^*\alpha$-$Cl[f^{-1}(V)]\subseteq f^{-1}[Ker(V)]=f^{-1}(V)$ and $s^*\alpha$-$Cl[f^{-1}(V)]=f^{-1}(V)$. This shows that $f^{-1}(V)$ is semi$^*\alpha$-closed in (X,τ).

Theorem 5.9. If a function $f:(X,\tau)\to(Y,\sigma)$ is contra semi$^*\alpha$-continuous and $g:(Y,\sigma)\to(Z,\eta)$ is continuous, then the composition $gof:(X,\tau)\to(Z,\eta)$ is contra semi$^*\alpha$-continuous.

Proof. Let W be any open set in (Z,η). Since, g is continuous, then $g^{-1}(W)$ is open in (Y,σ) and since f is contra semi$^*\alpha$-continuous, then $f^{-1}[g^{-1}(W)]=(gof)^{-1}(W)$ is semi$^*\alpha$-closed in (X,τ). Therefore, gof is contra semi$^*\alpha$-continuous.

Theorem 5.10. If $f:(X,\tau)\to(Y,\sigma)$ is semi$^*\alpha$-irresolute and $g:(Y,\sigma)\to(Z,\eta)$ is contra α-continuous, then their composition $gof:(X,\tau)\to(Z,\eta)$ is contra semi$^*\alpha$-continuous.

Proof. Let W be any open set in (Z,η). Since, g is contra semi$^*\alpha$-continuous, then $g^{-1}(W)$ is semi$^*\alpha$-closed in (Y,σ) and since f is semi$^*\alpha$-irresolute, then $f^{-1}[g^{-1}(W)]=(gof)^{-1}(W)$ is semi$^*\alpha$-closed in (X,τ). Therefore, gof is contra semi$^*\alpha$-continuous.

Theorem 5.11. Let $f:(X,\tau)\to(Y,\sigma)$ be surjective semi$^*\alpha$-irresolute and semi$^*\alpha$-open function and let $g:(Y,\sigma)\to(Z,\eta)$ be any function. Then their composition $gof:(X,\tau)\to(Z,\eta)$ is contra semi$^*\alpha$-continuous if and only if g is contra semi$^*\alpha$-continuous.

Proof. The if part is easy to prove. To prove the only if part, let B be any closed set in (Z,η). Since gof is contra semi$^*\alpha$-continuous, then $(gof)^{-1}(B)=f^{-1}[g^{-1}(B)]$ is semi$^*\alpha$-open in (X,τ) and since f is semi$^*\alpha$-open surjection, then $f[f^{-1}(g^{-1}(B))]=g^{-1}(B)$ is semi$^*\alpha$-open in (Y,σ). Therefore, g is contra semi$^*\alpha$-continuous.

Definition 5.12. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be pre semi$^*\alpha$-open if the image of every semi$^*\alpha$-open set of X is semi$^*\alpha$-open in Y.

Theorem 5.13. Let $f:(X,\tau)\to(Y,\sigma)$ be surjective, semi$^*\alpha$-irresolute and pre semi$^*\alpha$-open function. Let $g:(Y,\sigma)\to(Z,\eta)$ be any map. Then $gof:(X,\tau)\to(Z,\eta)$ is contra semi$^*\alpha$-continuous if and only if g is contra semi$^*\alpha$-continuous.

Proof. The if part is easy to prove. To prove the "only if" part, let $gof:(X,\tau)\to(Z,\eta)$ be contra semi*α-continuous function and let B be a closed subset of Z. Then $(gof)^{-1}(B)$ is semi*α-open in X which implies $f^{-1}\left[g^{-1}(B)\right]$ is semi*α-open in X. Since, f is pre semi*α-open, $f\left[f^{-1}\left(g^{-1}(B)\right)\right]=g^{-1}(B)$ is semi*α-open in Y. So, $g^{-1}(B)$ is semi*α-open in Y. Therefore, g is contra semi*α-continuous.

Theorem 5.14. If a function $f:(X,\tau)\to(Y,\sigma)$ is contra semi*α-continuous and Y is regular, then f is semi*α-continuous.

Proof. Let $x\in X$ and V be an open set in Y with $f(x)\in V$. Since Y is regular, there exists an open set W in Y such that $f(x)\in W\subseteq Cl(W)\subseteq V$. Since f is contra semi*α-continuous, by Theorem 5.8, there exists an semi*α-open U in X with $x\in U$ such that $f(U)\subseteq Cl(W)$. Then $f(U)\subseteq Cl(W)\subseteq V$. Hence f is semi*α-continuous.

Definition 5.15. Let $f:(X,\tau)\to(Y,\sigma)$ be a function. Then the subset $Gr(f)=\{(x,f(x)):x\in X\}\subseteq X\times Y$ is called the graph of f.

Theorem 5.16. A function $f:(X,\tau)\to(Y,\sigma)$ is contra semi*α-continuous if and only if the graph function $g:X\to X\times Y$ defined by $g(x)=(x,f(x))$ for each $x\in X$ is contra semi*α-continuous.

Proof. Necessity: Let f be contra semi*α-continuous. Now let $x\in X$ and let W be a closed subset of $X\times Y$ containing $g(x)$. Then $W\cap(\{x\}\times Y)$ is closed in $\{x\}\times Y$ containing $g(x)$. Also $\{x\}\times Y$ is homeomorphic to Y. Hence $\{y\in Y:(x,y)\in W\}$ is a closed subset of Y. Since f is contra semi*α-continuous, $\cup\{f^{-1}(y):(x,y)\in W\}$ is an semi*α-open subset of (X,τ). Further, $x\in\cup\{f^{-1}(y):(x,y)\in W\}\subseteq g^{-1}(W)$. Hence $g^{-1}(W)$ is semi*α-open. Then g is contra semi*α-continuous.

Sufficiency: Suppose that g is contra semi*α-continuous and let B be a closed subset of Y. Then $X\times B$ is a closed subset of $X\times Y$. Since g is contra semi*α-continuous, $g^{-1}(X\times B)$ is an semi*α-open subset of X. Also, $g^{-1}(X\times B)=f^{-1}(B)$. Hence f is contra semi*α-continuous.

Theorem 5.17. Let $\{(X_i,\tau_i):i\in I\}$ be any family of topological spaces. If $f:(X,\tau)\to\prod X_i$ is a contra semi*α-continuous function, then $\pi_i of:(X,\tau)\to(X_i,\tau_i)$ is contra semi*α-continuous, for each $i\in I$, where π_i is the projection of $\prod X_i$ onto X_i.

Proof. Suppose U_i is an arbitrary open set in X_i for each $i\in I$. Then $\pi_i^{-1}(U_i)$ is open in $\prod X_i$. Since f is contra semi*α-continuous, $f^{-1}\left[\pi_i^{-1}(U_i)\right]=(\pi_i of)^{-1}(U_i)$ is semi*α-closed in X. Therefore, $\pi_i of$ is contra semi*α-continuous.

Definition 5.18. Let X be a vector space over the field $K=$ ¡ *or* £ with standard topology. Let τ be a topology on X such that the following conditions are satisfied:

(1). For each $x,y\in X$ and each open set $W\subseteq X$ containing $x+y$, there exist open sets U and V in X containing x and y respectively, such that $U+V\subseteq W$,

(2). For each $\lambda\in$ K, $x\in X$ and each open set $W\subseteq X$ containing λx, there exist open sets U in K containing λ and V in X containing x such that $U.V\subseteq W$.

Then the pair $(X_{(K)},\tau)$ is called a topological vector space.

Theorem 5.19. Every topological vector space $\left(X_{(K)},\tau\right)$ is a regular topological space: i.e., If A is a closed subset of X and x is an element of X that is not in A, then there exist disjoint open sets U and V such that $x\in U$ and $A\subseteq V$. Equivalently, for each point $x\in X$ and each open subset U of X containing x, there exists an open subset V of X such that $x\in V\subseteq Cl(V)\subseteq U$.

Definition 5.20. A contra semi*α-topological vector space $\left(X_{(K)},\tau\right)$ is a vector space X over the field K ($\mathbb{R}$ or $\mathbb{C}$) with a topology τ defined on $X_{(K)}$ and standard topology on K such that:

(1) for each $x,y\in X$, and for each closed set $W\subseteq X$ containing $x+y$, there exist semi*α-open neighborhoods U and V of x and y, respectively in X such that $U+V\subseteq W$

(2) for each $\lambda\in K$, $x\in X$, and for each closed neighborhood $W\subseteq X$ containing $\lambda.x$, there exist semi*α-open open neighborhoods U of λ in K and V of x in X such that $U.V\subseteq W$.

Remark 5.21. It follows from the definition that topological vector space and contra semi*α-topological vector spaces are independent concepts.

Theorem 5.22. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological vector space. Suppose $T_x:X\to X$ is a right translation and $M_\lambda:X\to X$ is multiplication mapping, then both T_x and M_λ are contra semi*α-continuous.

Proof. Let y be an arbitrary element in X and let W be a closed neighborhood of $T_x(y)=y+x$. By definition of contra semi*α-topological vector space, there exist semi*α-open neighborhoods U and V containing y and x, respectively, such that $U+V\subseteq W$. In particular, we have $U+x\subseteq W$ which means $T_x(U)\subseteq W$. The inclusion shows that T_x is contra semi*α-continuous at y. Hence T_x is contra semi*α-continuous on X. Now we prove the statement for multiplication mapping. Let $\lambda\in K$ and $x\in X$. Let W be a closed neighborhood of $M_\lambda(x)=\lambda.x$. By definition of contra semi*α-topological vector spaces, there exist semi*α-open neighborhoods U in K and V in X containing λ and x, respectively such that $U.V\subseteq W$. In particular, we have $\lambda.V\subseteq W$, which means $M_\lambda(V)\subseteq W$. The inclusion shows that M_λ is contra semi*α-continuous at x. Hence M_λ is contra semi*α-continuous on X.

Theorem 5.23. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological topological vector space. Let A be a closed subset of X. Then the following statements are true:

(1) For every $y\in X$, $A+y\in s^*\alpha\text{-}O(X,\tau)$.

(2) For every nonzero $\lambda\in K$, $\lambda.A\in s^*\alpha\text{-}O(X,\tau)$.

Proof. (1). Let $z\in A+y$. We have to show that z is a semi*α-interior point of $A+y$. Now $z=x+y$, where x is some point in A. Then $T_{-y}:X\to X$ is contra semi*α-continuous at $z\in X$. Thus, for the closed set A containing x; $T_{(-y)}(z)=x$, there exists a semi*α-open neighborhood G_z of z such that $T_{(-y)}(G_z)=G_z+(-y)\subseteq A$. This implies $G_z\subseteq A+y$ which shows that z is a semi*α-interior point of $A+y$. Hence $A+y\in s^*\alpha\text{-}O(X,\tau)$.

(2) Let $z\in\lambda.A$. We have to show that z is an semi*α-interior point of $\lambda.A$. Now $z=\lambda.x$ for some x in A. We have multiplication mapping $M_{\lambda-1}:X\to X$ is contra semi*α-continuous. Thus, for the closed set A containing

$M_{\lambda^{-1}}(z)=\lambda^{-1}.z=x,$ there exists a semi$^*\alpha$-open neighborhood H_z of z in X such that $M_{\lambda^{-1}}(H_z)=\lambda^{-1}.H_z\subseteq A.$ This implies $H_z\subseteq\lambda.A.$ This shows that z is a semi$^*\alpha$-interior point of $\lambda.A.$ Hence $\lambda.A\in\alpha O(X,\tau).$

An immediate consequence of Theorem 5.23 we obtain the following result.

Theorem 5.24. For any closed subset A of a contra semi*α-topological vector space $X,$ the following assertions are true:

(a) $x+A\subseteq Int\left[Cl\left(Int(x+A)\right)\right]$ for each $x\in X.$

(b) $\lambda A\subseteq Int\left[Cl\left(Int(\lambda A)\right)\right]$ for each non-zero scalar $\lambda.$

Theorem 5.25. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological vector space. Let A be a closed subset of X and B be any subset of $X.$ Then $A+B\in s^*\alpha\text{-}O(X,\tau).$

Proof. By Theorem 5.23, we have $A+b\in s^*\alpha\text{-}O(X,\tau),$ for any $b\in B.$ Then it follows that $A+B=\bigcup_{b\in B}(A+b)$ is semi$^*\alpha$-open in $X.$

Corollary 5.26. Suppose $\left(X_{(K)},\tau\right)$ is a contra semi*α-topological vector space and A be a closed subset of $X.$ Then the set $U=\bigcup_{n=1}^{\infty}nA\in\alpha O(X,\tau).$

Theorem 5.27. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological topological vector space. Let G be an open subset of $X.$ Then the following statements are true:

(a) $x+G\in s^*\alpha\text{-}C(X,\tau)$ for each $x\in X.$

(b) $\lambda.G\in s^*\alpha\text{-}C(X,\tau)$ for each non-zero scalar $\lambda.$

Proof. (a) Suppose that $y\in s^*\alpha\text{-}Cl(x+G).$ Consider $z=-x+y.$ Then we need to prove that $y\in x+G.$ Assume that $y\notin x+G.$ This implies $z=-x+y\notin G.$ Then $z\in W=X-G$ and W is closed in $X.$ Sine X is contra semi*α-topological vector space. Therefore, there exist semi$^*\alpha$-open sets U and V in X such that $-x\in U,$ $y\in V$ and $U+V\subseteq W.$ Since $y\in s^*\alpha\text{-}Cl(x+G),$ $(x+G)\cap V\neq\phi.$ So there is $a\in(x+G)\cap V.$ Now $-x+a\in G\cap(U+V)\subseteq G\cap W=G\cap(X-G).$ Thus $-x+a\in G\cap(X-G)=\phi.$ But this is impossible. Therefore $y\in x+G$ and we conclude that $s^*\alpha\text{-}Cl(x+G)\subseteq x+G.$ Hence $x+G\in s^*\alpha\text{-}C(X,\tau).$

(b) Suppose that $x\in s^*\alpha\text{-}Cl(\lambda G).$ Then we wat to prove that $x\in\lambda G.$ Suppose that $x\notin\lambda G.$ Then $y=\frac{1}{\lambda}x\notin G.$ Hence $y=\frac{1}{\lambda}x\in W=X-G$ and W is closed. Since X is contra semi*α-topological topological vector space, there exist semi$^*\alpha$-open sets U in K containing $\frac{1}{\lambda}$ and V in X containing x such that $U.V\subseteq W.$ By hypothesis $x\in s^*\alpha\text{-}Cl(\lambda G),$ we have $(\lambda G)\cap V\neq\phi.$ Therefore, there is $a\in(\lambda G)\cap V.$ Now $z=\frac{1}{\lambda}a\in G\cap(U.V)\subseteq G\cap W=G\cap(X-G).$ Hence $z\in G\cap(X-G)=\phi.$ But this is impossible. Hence $x\in\lambda G$ and we conclude that $s^*\alpha\text{-}Cl(\lambda G)\subseteq\lambda G.$ Thus $\lambda G\in s^*\alpha\text{-}C(X).$

An immediate consequence of Theorem 5.27 follows in the next Theorem.

Theorem 5.28. For any open subset G of a contra semi*α-topological vector space X, the following assertions are true:

(a) $Cl\left[Int\left(Cl\left(x+G\right)\right)\right]\subseteq x+G$ for each $x\in X$.

(b) $Cl\left[Int\left(Cl\left(\lambda G\right)\right)\right]\subseteq \lambda G$ for each non-zero scalar λ.

Theorem 5.29. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological topological vector space. Let A be a subset of X and $x\in X$. Then the following statements are true:

(a) $s^{*}\alpha\text{-}Cl\left(x+A\right)\subseteq x+Cl\left(A\right)$.

(b) $x+s^{*}\alpha\text{-}Cl\left(A\right)\subseteq Cl\left(x+A\right)$.

(c) $x+Int\left(A\right)\subseteq s^{*}\alpha\text{-}Int\left(x+A\right)$.

(d) $Int\left(x+A\right)\subseteq x+s^{*}\alpha\text{-}Int\left(A\right)$.

Proof. (a) Let $y\in s^{*}\alpha\text{-}Cl\left(x+A\right)$ and consider $z=-x+y$ in X. Let W be any open neighborhood of z. Since $\left(X_{(K)},\tau\right)$ being a topological vector is regular. So there exists an open set G such that $z\in G\subseteq Cl\left(G\right)\subseteq W$. Since X is contra semi*α-topological vector space. So there exist semi*α-open sets U containing $-x$ and V containing y in X such that $U+V\subseteq Cl\left(G\right)$. By assumption, we have $\left(x+A\right)\cap V\neq\phi$, so there is $a\in X$ such that $a\in\left(x+A\right)\cap V$. It follows that $-x+a\in A\cap\left(U+V\right)\subseteq A\cap Cl\left(G\right)\subseteq A\cap W$. This implies $A\cap W\neq\phi$ and hence we obtain $z=-x+y\in Cl\left(A\right)$, that is, $y\in x+Cl\left(A\right)$. Therefore, $s^{*}\alpha\text{-}Cl\left(x+A\right)\subseteq x+Cl\left(A\right)$.

(b) Let $z\in x+s^{*}\alpha\text{-}Cl\left(A\right)$. Then $z=x+y$ for some $y\in s^{*}\alpha\text{-}Cl\left(A\right)$. Let W be any open neighborhood of z. Since $\left(X_{(K)},\tau\right)$ being a topological vector is regular. So there exists an open set G in X such that $z\in G\subseteq Cl\left(G\right)\subseteq W$. Since X is contra semi*α-topological vector space. So there exist semi*α-open sets $U,V\in s^{*}\alpha\text{-}O\left(X\right)$ such that $x\in U$, $y\in V$ and $U+V\subseteq Cl\left(G\right)$. Since $y\in s^{*}\alpha\text{-}Cl\left(A\right)$, so we have $A\cap V\neq\phi$. This implies there is $a\in A\cap V$. Thus $x+a\in\left(x+A\right)\cap\left(U+V\right)\subseteq\left(x+A\right)\cap Cl\left(G\right)\subseteq\left(x+A\right)\cap W$. This implies $\left(x+A\right)\cap W\neq\phi$. Thus $z=x+y\in Cl\left(x+A\right)$. Hence the assertion follows.

(c) Let $y\in x+Int\left(A\right)$. Then $z=-x+y\in Int\left(A\right)$. Since $\left(X_{(K)},\tau\right)$ being a topological vector is regular. So there exists an open set G in X such that $z\in G\subseteq Cl\left(G\right)\subseteq Int\left(A\right)$. Now $z\in Cl\left(G\right)$ and X is contra semi*α-topological vector space. So there exist semi*α-open sets $U,V\in s^{*}\alpha\text{-}O\left(X\right)$ such that $-x\in U$, $y\in V$ and $U+V\subseteq Cl\left(G\right)$. This implies $z=-x+y\in -x+V\subseteq U+V\subseteq Cl\left(G\right)\subseteq A$. Whence we obtain $y\in V\subseteq x+A$. Since V is semi*α-open, so it shows that $y\in s^{*}\alpha\text{-}Int\left(x+A\right)$ and hence consequently, $x+Int\left(A\right)\subseteq s^{*}\alpha\text{-}Int\left(x+A\right)$.

(d) Let $z=x+y\in Int\left(x+A\right)$ for some $y\in A$. Since $\left(X_{(K)},\tau\right)$ being a topological vector is regular. So there exists an open set G in X such that $z\in G\subseteq Cl\left(G\right)\subseteq Int\left(x+A\right)$. Now $z\in Cl\left(G\right)$ and X is contra semi*α-topological vector space. Hence there exist semi*α-open sets $U,V\in s^{*}\alpha\text{-}O\left(X\right)$ such that $x\in U$, $y\in V$ and $U+V\subseteq Cl\left(G\right)\subseteq Int\left(x+A\right)\subseteq x+A$. Hence $z=x+y\in x+V\subseteq U+V\subseteq Cl\left(G\right)\subseteq Int\left(x+A\right)\subseteq x+A$. This implies that $y\in V\subseteq A$. Since V is semi*α-open, so it shows that $y\in s^{*}\alpha\text{-}Int\left(A\right)$ and hence $z=x+y\in x+s^{*}\alpha\text{-}Int\left(A\right)$, consequently, we conclude that $Int\left(x+A\right)\subseteq x+s^{*}\alpha\text{-}Int\left(A\right)$.

Corollary 5.30. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological vector space. Let A be a subset of X and $x\in X$. Then the following statements are true:

(a) $s^*\alpha\text{-}Cl(x+A)\subseteq x+Cl_\theta(A)$.

(b) $x+s^*\alpha\text{-}Cl(A)\subseteq Cl_\theta(x+A)$.

(c) $x+Int_\theta(A)\subseteq s^*\alpha\text{-}Int(x+A)$.

(d) $Int_\theta(x+A)\subseteq x+s^*\alpha\text{-}Int(A)$.

Proof. We notice that for any topological space (X,τ) we have $\theta O(X,\tau)\subseteq\tau$. Hence for any subset A of X, $Cl(A)\subseteq Cl_\theta(A)$ and $Int(A)_\theta\subseteq Int(A)$. Then by using Theorem 5.29, the results follow immediately.

Similarly, arguments as in the preceding Theorem 5.29, we obtain the following useful result.

Theorem 5.31. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological vector space. Let A be a subset of X and λ any non-zero scalar in K. Then the following assertions are true:

(a) $s^*\alpha\text{-}Cl(\lambda.A)\subseteq\lambda.Cl(A)$.

(b) $\lambda.s^*\alpha\text{-}Cl(A)\subseteq Cl(\lambda.A)$.

(c) $\lambda.Int(A)\subseteq s^*\alpha\text{-}Int(\lambda.A)$.

(d) $Int(\lambda.A)\subseteq s^*\alpha\text{-}.\alpha Int(A)$.

Theorem 5.32. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological vector space. Then the mapping $\Phi: X\times X\to X$ defined by $\Phi(x,y)=x+y,\ \forall\,(x,y)\in X\times X$ is contra semi*α-continuous.

Proof. Let $(x,y)\in X\times X$ and let W be a closed set in X such that $\Phi(x,y)=x+y\in W$. Then, by the definition of contra semi*α-topological vector space, there exist semi*α-open sets $U,V\in s^*\alpha\text{-}O(X)$ such that $x\in U,\ y\in V$ and $U+V\subseteq W$. Since $U\times V$ is α-open in $X\times X$ (with respect to the product topology) such that $\Phi(U\times V)=U+V\subseteq W$. Hence, Φ is contra semi*α-continuous at arbitrary point $x\times y\in X\times X$. Therefore, Φ is contra semi*α-continuous.

Theorem 5.33. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological vector space. Then the mapping $\Psi: K\times X\to X$ defined by $\Psi(\lambda,x)=\lambda.x,\ \forall\,(\lambda,x)\in K\times X$ is contra semi*α-continuous.

Proof. Trivial proof is omitted.

Theorem 5.34. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological vector space and $\left(Y_{(K)},\sigma\right)$ be a topological vector space. Let $f: X\to Y$ be a linear map such that f is contra semi*α-continuous at 0. Then f is contra semi*α-continuous everywhere.

Proof. Let x be any non-zero element of X and V be a closed set in Y containing $f(x)$. Since translation of a closed set in topological vector space is closed, $V-f(x)$ is closed set in Y containing 0. Since f is contra semi*α-continuous at 0, there exists an α-open set U in X containing 0 such that $f(U)\subseteq V-f(x)$. Furthermore,

linearity of f implies that $f(x+U)\subseteq V$ and $x+U$ is semi*α-open set in X. Hence f is contra semi*α-continuous at x. By hypothesis, f is contra semi*α-continuous at 0. This reflects that f is contra semi*α-continuous.

Theorem 5.35. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological vector space. Let $f:X\to K$ be a linear functional which is contra semi*α-continuous at 0. Then the set $E=\{x\in X: f(x)=0\}$ is semi*α-open set in X.

Proof. The easy proof is omitted.

Theorem 5.36. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological vector space and A be any semi*α-compact subset of X. Then $y+A$ is compact, for each $y\in X$.

Proof. Let $\xi\subseteq\tau$ such that $y+A\subseteq\cup\xi$. Therefore for each $x\in A$, there exists $U_x\in\xi$ such that $y+x\in U_x$. Since X being a topological vector space is regular. Hence there exists $V_x\in\tau$ such that $y+x\in V_x\subseteq Cl(V_x)\subseteq U_x$. This implies $x\in -y+V_x\subseteq -y+Cl(V_x)\subseteq -y+U_x$, for each $x\in A$. Then by Theorem 5.23, $\xi^*=\{-y+Cl(V_x): x\in A\}$ $\subseteq s^*\alpha\text{-}O(X,\tau)$ and clearly $A\subseteq\cup\xi^*$. By hypothesis, A is semi*α-compact. Therefore $A\subseteq\cup\{-y+Cl(V_x): x\in B\}$ for some finite subset $B\subseteq A$. Whence we find that $y+A\subseteq\cup\{Cl(V_x): x\in B\}$. Thus $y+A\subseteq\cup\{U_x: x\in B\}$. This shows that $y+A$ is compact. Hence the proof.

Theorem 5.37. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological vector space and A be any semi*α-compact subset of X. Then λA is compact, for each scaler λ.

Proof. If $\lambda=0$ we are nothing to prove. Assume that λ is non-zero. Let $\xi\subseteq\tau$ such that $\lambda A\subseteq\cup\xi$. Therefore for each $x\in A$, there exists $U_x\in\xi$ such that $\lambda x\in U_x$. Since X being a topological vector space is regular. Hence there exists $V_x\in\tau$ such that $\lambda x\in V_x\subseteq Cl(V_x)\subseteq U_x$. This implies that $x\in\frac{1}{\lambda}V_x\subseteq\frac{1}{\lambda}Cl(V_x)\subseteq\frac{1}{\lambda}U_x$. Then by Theorem 5.23, it follows that $\xi^*=\left\{\frac{1}{\lambda}Cl(V_x): x\in A\right\}\subseteq s^*\alpha\text{-}O(X,\tau)$ and clearly $A\subseteq\cup\xi^*$. By hypothesis, A is a semi*α-compact subset of X. Thus $A\subseteq\cup\left\{\frac{1}{\lambda}Cl(V_x): x\in B\right\}$ for some finite subset $B\subseteq A$. Whence we find that $\lambda A\subseteq\cup\{Cl(V_x): x\in B\}$. This implies that $\lambda A\subseteq\cup\{U_x: x\in B\}$. This shows that λA is compact.

Definition 5.38. A function $f:(X,\tau)\to(Y,\sigma)$ is said to be contra semi$^*\alpha$-open if $f(U)$ is semi$^*\alpha$-closed in (Y,σ) for every open set U of (X,τ).

Theorem 5.39. The following statements are equivalent for function $f:(X,\tau)\to(Y,\sigma)$:

(1) f is contra semi$^*\alpha$-open;

(2) $f(U)$ is semi$^*\alpha$-open in Y for every closed set U in X;

(3) For every subset B of Y and for every open set F of X with $f^{-1}(B)\subseteq F$, there exists an semi$^*\alpha$-closed set V of Y with $B\subseteq V$ and $f^{-1}(V)\subseteq F$;

(4) For each $y\in Y$ and every open set F of X with $f^{-1}(y)\subseteq F$, there exists a semi$^*\alpha$-closed set V of Y with $y\in V$ and $f^{-1}(V)\subseteq F$.

Definition 5.40. A bijective mapping f from a topological space to itself is called contra semi*α-homeomorphism if it is contra semi*α-continuous and contra semi*α-open.

Definition 5.41. A contra semi*α-topological vector space $\left(X_{(K)},\tau\right)$ is said to be contra semi*α-homogeneous space if for all $x,y\in X,$ there is a contra semi*α-homeomorphism f of the space X onto itself such that $f(x)=y.$

Theorem 5.42. Let $\left(X_{(K)},\tau\right)$ be a contra semi*α-topological vector space. For given $y\in X$ and λ in K with $\lambda\neq 0,$ the right (left) translation map $T_y : x\to x+y$ and multiplication map $M_\lambda : x\to \lambda.x,$ where $x\in X$ are contra semi*α-homeomorphisms onto itself.

Proof. It is obvious that right translations are bijective mappings. Then the translations T_y and M_λ are contra semi*α-continuous mappings. We prove that the translation T_y is contra semi*α-open. Let U be any closed neighborhood of $x.$ Then $T_y(U)=U+y.$ By Theorem 5.23, $U+y$ is semi*α-open in $X.$ This proves that T_y is contra semi*α-open mapping. Thus, we conclude that T_y is contra semi*α-homeomorphism. Similarly, we can prove that $M_\lambda : x\to \lambda.x,$ is a contra semi*α-homeomorphism.

Theorem 5.43. Every contra semi*α-topological vector space $\left(X_{(K)},\tau\right)$ is a contra semi*α-homogeneous space.

Proof. Take any $x,y\in X$ and put $z=(-x)+y.$ Then $T_z : X\to X$ is an semi*α-homeomorphism and $T_z(x)=x+z=y.$

Theorem 5.44. Suppose $\left(X_{(K)},\tau\right)$ is a contra semi*α-topological vector space and S is a subspace of $X.$ If S contains a nonempty closed set, then S is semi*α-open in $X.$

Proof. Suppose $U\neq\phi$ is a closed subset of X such that $U\subseteq S.$ For any $y\in S$ the set $T_y(U)=U+y$ is semi*α-open in X and is a subset of $S.$ Therefore, the subspace $S=\bigcup\{U+y : y\in S\}$ is semi*α-open in $X,$ as being the union of semi*α-open sets.

Theorem 5.45. Every closed subspace S of a contra semi*α-topological vector space $\left(X_{(K)},\tau\right)$ is also a contra semi*α-topological vector space (called contra semi*α-topological subspace of X).

Proof. Let $x,y\in S$ and W be a closed neighborhood of $x+y$ in $S.$ This gives W is a closed neighborhood of $x+y$ in $X.$ Hence there exist semi*α-open neighborhoods $U\subseteq X$ of x and $V\subseteq X$ of y such that $U+V\subseteq W.$ Then the sets $A=U\cap S$ and $B=V\cap S$ are semi*α-open neighborhoods of x and $y,$ respectively in $S.$ Also $A+B\subseteq U+V\subseteq W.$ Again, let $\lambda\in K$ and $x\in S.$ Let W be a closed neighborhood of $\lambda.x$ in $S.$ Since S is closed in $X,$ W is a closed neighborhood of $\lambda.x$ in $X.$ Hence there exist semi*α-open neighborhoods $U\subseteq K$ of λ and $V\subseteq X$ of x such that $U.V\subseteq W.$ Then the set $A=U\cap K$ is semi*α-open neighborhood of λ in K and the set $B=V\cap S$ is semi*α-open neighborhood of y in $S.$ Also $A.B\subseteq U.V\subseteq W,$ which means that S is a contra semi*α-topological vector space.

Theorem 5.46. In a contra semi*α-topological vector space, for any closed neighborhood U of $e,$ there is an semi*α-open neighborhood of e such that $V+V\subseteq U.$

Proof. Proof is simple, therefore omitted.

Theorem 5.47. Let A and B be subsets of a contra semi*α-topological vector space $\left(X_{(K)}, \tau\right)$. Then $s^*\alpha\text{-}Cl(A)+s^*\alpha\text{-}Cl(B)\subseteq ker(A+B)$.

Proof. Suppose that $z=x+y$, where $x\in s^*\alpha\text{-}Cl(A)$, $y\in s^*\alpha\text{-}Cl(B)$. Let W be a closed neighborhood of $z=x+y$. Then there are semi*α-open neighborhoods U and V of x and y, respectively such that $U+V\subseteq W$. Since $x\in s^*\alpha\text{-}Cl(A)$, $y\in s^*\alpha\text{-}Cl(B)$, there are $a\in A\cap U$ and $b\in B\cap V$. Then $a+b\in(A+B)\cap(U+V)\subseteq (A+B)\cap W$ This implies $(A+B)\cap W\neq\phi$. Therefore, it shows that $z=x+y\in ker(A+B)$, that is $s^*\alpha\text{-}Cl(A)+s^*\alpha\text{-}Cl(B)\subseteq ker(A+B)$.

6. SEPARATION AXIOMS WITH RESPECT TO SEMI-STAR-ALPHA-OPEN-SETS

In this section, we utilize *semi*α-open* sets to introduce the concepts of *semi*α-regular*, *semi*α-normal* and $s^*\alpha\text{-}T_i$*-spaces* $(i=0,1,2,3,4)$. We give some characterizations for each one of them and elucidate the relationships among themselves as well as with $s^*\alpha\text{-}T_i$*-space.*

Definition 6.1. A topological space (X,τ) is said to be:

(i) $s^*\alpha\text{-}T_0$ if for every $a\neq b\in X$, there exists a $s^*\alpha$*-open* set containing only one of them.

(ii) $s^*\alpha\text{-}T_1$ if for every $a\neq b\in X$, there exist two *semi*α-open* sets one of them contains a but not b and the other contains b but not a.

(iii) *semi*α-Hausdorff* $\left(or\ s^*\alpha\text{-}T_2\right)$ if for every $a\neq b\in X$, there exist two disjoint *semi*α-open* sets U and V containing a and b, respectively.

(iv) *semi*α-regular* if for every *semi*α-closed* set F and each $a\notin F$, there exist disjoint *semi*α-open* sets U and V containing F and a, respectively.

(v) *semi*α-normal* if for every disjoint *semi*α-closed* sets F and H, there exist disjoint *semi*α-open* sets U and V containing F and H, respectively.

(vi) $s^*\alpha\text{-}T_3$ $\left(resp.\ s^*\alpha\text{-}T_4\right)$ if it is both *semi*α-regular* $\left(resp.\ semi^*\alpha\text{-}normal\right)$ and $s^*\alpha\text{-}T_1$.

Theorem 6.2. The following three statements are equivalent:

(1) (X,τ) is an $s^*\alpha\text{-}T_0$*-space.*

(2) $s^*\alpha\text{-}Cl(\{a\})\neq s^*\alpha\text{-}Cl(\{b\})$ for each $a\neq b\in X$.

(3) For each $a\in X$, we have $s^*\alpha\text{-}d(\{a\})$ is a union of *semi*α-closed* sets.

Proof. $(1)\Rightarrow(2)$: For each $a\neq b\in X$, there exists a *semi*α-open* set G containing a but not b, or containing b but not a. Say $a\in G$ and $b\notin G$. Then $a\notin s^*\alpha\text{-}cl(\{b\})$ because G is a *semi*α-open* set containing a such that $G\cap\{b\}=\phi$. Since $a\in s^*\alpha\text{-}cl(\{b\})$, then $s^*\alpha\text{-}cl(\{a\})\neq s^*\alpha\text{-}cl(\{b\})$.

$(2)\Rightarrow(3)$: Let $b\in s^*\alpha\text{-}d(\{a\})$. Then $b\neq a$ and $b\in\{a\}\cup s^*\alpha\text{-}d(\{a\})=s^*\alpha\text{-}Cl(\{a\})$. Thus, $s^*\alpha\text{-}Cl(\{b\})\subseteq s^*\alpha\text{-}Cl(\{a\})$.

So $b \in s^*\alpha\text{-}Cl(\{b\}) \subseteq s^*\alpha\text{-}d(\{a\})$. Hence, $s^*\alpha\text{-}d(\{a\}) = \cup\{s^*\alpha\text{-}Cl(\{b\}) : \text{for each } b \in s^*\alpha\text{-}d(\{a\})\}$.

$(3) \Rightarrow (1)$: Let $a \neq b$. Then we have two cases:

(i) Either $b \in s^*\alpha\text{-}d(\{a\})$. Then there is a *semi*$^*\alpha$*-closed* set F such that $b \in F \subseteq s^*\alpha\text{-}d(\{a\})$. Since $a \notin s^*\alpha\text{-}d(\{a\})$, then $a \notin F$. Therefore F^c is a *semi*$^*\alpha$*-open* set containing a such that $b \notin F^c$.

(ii) Or $b \notin s^*\alpha\text{-}d(\{a\})$. Then there is a *semi*$^*\alpha$*-open* set G containing b such that $a \notin G$.

In the above both cases, we infer that (X, τ) is an $s^*\alpha\text{-}T_0$*-space*.

Corollary 6.3. An $s^*\alpha\text{-}T_0$*-space* (X, τ) contains at most one *semi*$^*\alpha$*-dense* singleton set ($\{a\}$ is a *semi*$^*\alpha$*-dense* set if $s^*\alpha\text{-}Cl(\{a\}) = X$).

Proof. Let (X, τ) be an $s^*\alpha\text{-}T_0$*-space*. Suppose that there are two distinct singleton sets $\{a\}$ and $\{b\}$ such that $s^*\alpha\text{-}Cl(\{a\}) = s^*\alpha\text{-}Cl(\{b\}) = X$. Then (X, τ) is not an $s^*\alpha\text{-}T_0$*-space*, a contradiction. Hence, (X, τ) contains at most one *semi*$^*\alpha$*-dense* singleton set.

Theorem 6.4. Let (X, τ) be a topological space. Then the following four statements are equivalent:

(1) (X, τ) is an $s^*\alpha\text{-}T_1$*-space*.

(2) Every singleton subset of (X, τ) is *semi*$^*\alpha$*-closed* set.

(3) The intersection of all *semi*$^*\alpha$*-open* sets containing a set A is exactly A.

(4) $s^*\alpha\text{-}d(\{a\}) = \phi$ for each $a \in X$.

Proof. $(1) \Rightarrow (2)$: Consider (X, τ) is an $s^*\alpha\text{-}T_1$*-space* and let $\{a\} \subseteq X$. For all $b \in X$ such that $a \neq b$, there exists a *semi*$^*\alpha$*-open* set G containing b such that $G \cap \{a\} = \phi$. Then $b \notin s^*\alpha\text{-}cl(\{a\})$. Therefore, $s^*\alpha\text{-}cl(\{a\}) = \{a\}$. Thus, $\{a\}$ is a *semi*$^*\alpha$*-closed* set.

$(2) \Rightarrow (3)$: Let A be a subset of (X, τ). Then for each $a \in A^c$, we have $\{a\}^c$ is an *semi*$^*\alpha$*-open* set containing A. Now, $A \subseteq \{G : G \text{ is an semi}^*\alpha\text{-open set containg } A\} \subseteq \{\{a\}^c : a \in A^c\} \subseteq A$. Therefore, it follows that $A = \{G : G \text{ is an semi}^*\alpha\text{-open set containg } A\}$, as required. $(3) \Rightarrow (4)$: Suppose that there exists $a \in X$ such that $s^*\alpha\text{-}d(\{a\}) \neq \phi$. Then there exists $b \neq a$ such that $b \in s^*\alpha\text{-}d(\{a\})$. Therefore $(G - \{b\}) \cap \{a\} \neq \phi$ for every *semi*$^*\alpha$*-open* set G containing b. This implies that any *semi*$^*\alpha$*-open* set containing b contains a as well. Thus the intersection of all *semi*$^*\alpha$*-open* sets containing b is not equal $\{b\}$. But this contradicts (3). Hence, it must be $s^*\alpha\text{-}d(\{a\}) = \phi$.

$(4) \Rightarrow (1)$: Let $a \neq b$. Since $s^*\alpha\text{-}d(\{a\}) = \phi$ and $s^*\alpha\text{-}d(\{b\}) = \phi$, then $\{a\}$ and $\{b\}$ are *semi*$^*\alpha$*-closed* sets. Therefore $\{a\}^c$ and $\{b\}^c$ are *semi*$^*\alpha$*-open* sets containing $\{b\}$ and $\{a\}$, respectively. Thus, (X, τ) is an $s^*\alpha\text{-}T_1$*-space*.

Proposition 6.5. Every (X, τ) satisfying the difference property for the collection of *semi*$^*\alpha$*-open* sets is an $s^*\alpha\text{-}T_1$*-space*.

Proof. Let $a \neq b \in X$, Since X is a $semi^*\alpha$-*open* set and (X,τ) satisfies the difference property for the collection of $semi^*\alpha$-*open* sets, then $X-\{a\}$ and $X-\{b\}$ are $semi^*\alpha$-*open* sets containing b and a, respectively, such that $a \notin X-\{a\}$ and $b \notin X-\{b\}$. Hence, (X,τ) is an $s^*\alpha$-T_1-*space*.

We need the following definition to obtain the equivalence between $s^*\alpha$-T_0 and $s^*\alpha$-T_1.

Definition 6.6. A topological space (X,τ) is called a $semi^*\alpha$-*symmetric* space if $a \in s^*\alpha\text{-}Cl(\{b\})$ implies that $b \in s^*\alpha\text{-}Cl(\{a\})$ for $a \neq b \in X$.

Theorem 6.7. Let (X,τ) be a $semi^*\alpha$-*symmetric* space. Then it is $s^*\alpha$-T_1 if and only if it is $s^*\alpha$-T_0.

Proof. The necessary condition is obvious.

To prove the sufficient condition, let $a \neq b$. Then there exists a $semi^*\alpha$-*open* set G containing only one of them. Say, $a \in G$ and $b \in G$. Therefore $a \notin s^*\alpha\text{-}Cl(\{b\})$. By the $semi^*\alpha$-*symmetry* of (X,τ), we have $b \notin s^*\alpha\text{-}Cl(\{a\})$. Thus $\left[s^*\alpha\text{-}Cl(\{a\})\right]^c$ is a $semi^*\alpha$-*open* set containing b. Hence, (X,τ) is $s^*\alpha$-T_1.

Theorem 6.8. Let (X,τ) be a topological space. Then the following three statements are equivalent:

(1) (X,τ) is an $s^*\alpha$-T_2-*space*.

(2) $\{a\} = \cap\{F_i : F_i \text{ is a } semi^*\alpha\text{-closed neighborhood of } a\}$ for each $a \in X$.

(3) The diagonal $\Delta = \{(a,a) : a \in X\}$ is $semi^*\alpha$-*closed* in the product space $X \times X$.

Proof. (1)$\Rightarrow$(2): Consider (X,τ) is an $s^*\alpha$-T_2-*space*. Then for $a \neq b$, there exist two disjoint $semi^*\alpha$-*open* sets G_i and H_i such that $a \in G_i$ and $b \in H_i$. Obviously, $G_i \subseteq H_i^c$. Therefore $a \in s^*\alpha\text{-}Cl(G_i) \subseteq H_i^c = F_i$. Thus, F_i is a $semi^*\alpha$-*closed* neighborhood of a such that $b \notin F_i$. So, $\{a\} = \cap\{F_i : F_i \text{ is a } semi^*\alpha\text{-closed neighborhood of } a\}$.

(2)$\Rightarrow$(1): Suppose that $a \neq b \in X$. Then by (2), $\{a\} = \cap\{F_i : F_i \text{ is a } semi^*\alpha\text{-closed neighborhood of } a\}$, hence, there exists a $semi^*\alpha$-*closed* neighborhood F_{i_0} of a such that $b \notin F_{i_0}$. Therefore there exists a $semi^*\alpha$-*open* set G containing a such that $s^*\alpha\text{-}Cl(G) \subseteq F_{i_0}$. It is clear that $\left[s^*\alpha\text{-}Cl(G)\right]^c$ is a $semi^*\alpha$-*open* set containing b and $G \cap \left[s^*\alpha\text{-}Cl(G)\right]^c = \phi$. Hence, (X,τ) is an $s^*\alpha$-T_2-*space*.

(1)$\Rightarrow$(3): Suppose that (X,τ) is an $s^*\alpha$-T_2-*space* and let $(a,b) \in X \times X - \Delta$. Then $a \neq b$. Therefore there exist two disjoint $semi^*\alpha$-*open* sets G and H containing a and b, respectively. Thus, $(a,b) \in G \times H \subseteq X \times X - \Delta$, proving that $X \times X - \Delta$ is a $semi^*\alpha$-*neighborhood* of any of its points. Hence, Δ is $semi^*\alpha$-*closed*.

(3)$\Rightarrow$(1): Suppose that Δ is a $semi^*\alpha$-*closed* subset of $X \times X$ and let $a \neq b \in X$. Then $X \times X - \Delta$ is a $semi^*\alpha$-*open* set containing (a,b). Therefore there exist two $semi^*\alpha$-*open* subsets G and H of (X,τ) such that $(a,b) \in G \times H \subseteq X \times X - \Delta$. This implies that G and H are two disjoint $semi^*\alpha$-*open* sets containing a and b, respectively. Hence, (X,τ) is an $s^*\alpha$-T_2-*space*.

Theorem 6.9. Let (X,τ) be a topological space. Then the following three statements are equivalent:

(1) (X,τ) is *semi*α-regular*.

(2) For each *semi*α-open* subset U of (X,τ) containing a, there exists a *semi*α-open* subset V of (X,τ) such that $a\in V\subseteq s^*\alpha\text{-}Cl(V)\subseteq U$.

(3) Every *semi*α-open* subset U of (X,τ) can be represented as follows:

$U=\bigcup\{H : H \text{ is a semi}^*\alpha\text{-open subset of } (X,\tau) \text{ and } s^*\alpha\text{-}Cl(H)\subseteq U\}$.

Proof. $(1)\Rightarrow(2)$: Let (X,τ) be a *semi*α-regular* space and U be a *semi*α-open* set such that $a\in U$. Then there exist disjoint *semi*α-open* sets V and W containing a and U^c, respectively. Therefore $a\in V\subseteq W^c\subseteq U$. Thus $a\in V\subseteq s^*\alpha\text{-}Cl(V)\subseteq U$.

$(2)\Rightarrow(3)$: Suppose that U is a *semi*α-open* set. By hypothesis, for each $a\in U$, there exists a *semi*α-open* set H such that $a\in H\subseteq s^*\alpha\text{-}Cl(H)\subseteq U$. Then $U=\bigcup\{H : H \text{ is semi}^*\alpha\text{-open and } s^*\alpha\text{-}Cl(H)\subseteq U\}$.

$(3)\Rightarrow(1)$: Let F be a *semi*α-closed* set such that $a\notin F$. Then $F^c=\bigcup\{H : H \text{ is semi}^*\alpha\text{-open and } s^*\alpha\text{-}Cl(H)\subseteq F^c\}$. Since $a\in F^c$, then there exists a *semi*α-open* set H_a containing a such that $s^*\alpha\text{-}Cl(H_a)\subseteq F^c$. Take $V=\left[s^*\alpha\text{-}Cl(H_a)\right]^c$. Then V is a *semi*α-open* set containing F and $V\cap H_a=\phi$. This completes the proof.

Theorem 6.10. Consider (X,τ) is a *semi*α-regular* space. Then the following statements are equivalent:

(1) (X,τ) is an $s^*\alpha\text{-}T_2$-*space*.

(2) (X,τ) is an $s^*\alpha\text{-}T_1$-*space*.

(3) (X,τ) is an $s^*\alpha\text{-}T_0$-*space*.

Proof. The implications $(1)\Rightarrow(2)\Rightarrow(3)$ are obvious.

$(3)\Rightarrow(1)$: Let $a\neq b\in X$. Since (X,τ) is an $s^*\alpha\text{-}T_0$-*space*, then from Theorem (6.2), we get $s^*\alpha\text{-}Cl(\{a\})\neq s^*\alpha\text{-}Cl(\{b\})$. Therefore $a\notin s^*\alpha\text{-}Cl(\{b\})$ or $b\notin s^*\alpha\text{-}Cl(\{a\})$. Say, $a\notin s^*\alpha\text{-}Cl(\{b\})$. Since (X,τ) is *semi*α-regular*, then there exist disjoint *semi*α-open* sets G and H containing a and $s^*\alpha\text{-}Cl(\{b\})$, respectively. Thus (X,τ) is an $s^*\alpha\text{-}T_2$-*space*.

Theorem 6.11. Let (X,τ) be a topological space. Then the following statements are equivalent:

(1) (X,τ) is *semi*α-normal*.

(2) For each *semi*α-closed* set F and each *semi*α-open* set U containing F, there exists a *semi*α-open* set V such that $F\subseteq V\subseteq s^*\alpha\text{-}Cl(V)\subseteq U$.

(3) For every *semi*α-open* sets U and V such that $U\cup V=X$, there are two *semi*α-closed* sets F and H contained

in U and V, respectively, such that $F \cup H = X$.

Proof. $(1) \Rightarrow (2)$: Consider (X, τ) is *semi*α-normal* and F is a *semi*α-closed* subset of a *semi*α-open* set U. Then U^c and F are disjoint *semi*α-closed* sets. Therefor there exist two disjoint *semi*α-open* sets W and V containing U^c and F, respectively. Thus $F \subseteq V \subseteq W^c = s^*\alpha\text{-}Cl(W^c) \subseteq U$. Hence, $F \subseteq V \subseteq s^*\alpha\text{-}Cl(V) \subseteq U$.

$(2) \Rightarrow (3)$: Consider U and V are *semi*α-open* sets such that $U \cup V = X$. Then U^c is a *semi*α-closed* set such that $U^c \subseteq V$. By (2), there is a *semi*α-open* set G such that $U^c \subseteq G \subseteq s^*\alpha\text{-}Cl(G) \subseteq V$. Thus $G^c \subseteq U$ and $s^*\alpha\text{-}Cl(G) \subseteq V$ are *semi*α-closed* sets such that $G^c \cup s^*\alpha\text{-}Cl(G) = X$.

$(3) \Rightarrow (1)$: Consider F and H are disjoint *semi*α-closed* sets. Since F^c and H^c are *semi*α-open* sets such that $F^c \cup H^c = X$, then there are two *semi*α-closed* sets M and N such that $M \subseteq F^c$, $N \subseteq H^c$ and $M \cup N = X$. Thus M^c and N^c are two disjoint *semi*α-open* sets containing F and H, respectively. Hence, (X, τ) is *semi*α-normal*.

Now, we show the implications of these separation axioms among themselves as well as with *s*α-T_i-space*.

Theorem 6.12. Every *s*α-T_i-space* is *s*α-T_{i-1}-space* for $i = 1, 2, 3, 4$.

We complete this section by discussing these separation axioms in terms of hereditary and topological properties and finite product space.

Definition 6.13. For a nonempty subset A of (X, τ), the family $\tau_A^{s^*\alpha} = \{A \cap G : G \text{ is semi}^*\alpha\text{-open subset of } (X, \tau)\}$ is called a relative *semi*α-topology* on A. A pair $\left(A, \tau_A^{s^*\alpha}\right)$ is called a *semi*α-subspace* of (X, τ). One can easily prove that a *semi*α-subspace* $\left(A, \tau_A^{s^*\alpha}\right)$ of (X, τ) is a supra topological space.

Proposition 6.14. Let $\left(Y, \tau_Y^{s^*\alpha}\right)$ be a *semi*α-subspace* of (X, τ). A subset H of Y is supra *semi*α-closed* in $\left(Y, \tau_Y^{s^*\alpha}\right)$ if and only if there exists a *semi*α-closed* subset F of (X, τ) such that $H = Y \cap F$.

Proof. Necessity: Let H be a supra *semi*α-closed* subset of $\left(Y, \tau_Y^{s^*\alpha}\right)$. Then there exists a supra *semi*α-open* subset W of $\left(Y, \tau_Y^{s^*\alpha}\right)$ such that $H = Y - W$. Now, there exists a *semi*α-open* subset V of (X, τ) such that $W = Y \cap V$. Therefore $H = Y - (Y \cap V) = Y \cap V^c$. By taking $F = V^c$, the proof of the necessary part is complete.

Sufficiency: Let $H = Y \cap F$ such that F is a *semi*α-closed* subset of (X, τ). Then $Y - H = Y - (Y \cap F) = (Y \cap X) - (Y \cap F) = Y \cap (X - F)$. Since $X - F$ is a *semi*α-open* subset of (X, τ), then $Y - H$ is a supra *semi*α-open* subset of $\left(Y, \tau_Y^{s^*\alpha}\right)$. Thus H is a supra *semi*α-closed* subset of $\left(Y, \tau_Y^{s^*\alpha}\right)$.

Definition 6.15. A property is said to be a relative *semi*α-heriditary* property if the property passes from a supra topological space *s*α-$O(X, \tau)$* to every relative *semi*α-subspace*.

Theorem 6.16. A property of being an *s*α-T_i-space* is a relative *semi*α-heriditary* for $i = 1, 2, 3$.

Proof. We shall suffice with proof of case $i=3$ which directly contains the case $i=1$. In a similar way, one can prove the cases $i=0, 2$.

Suppose that $\left(A, {}_{\tau}^{s^*\alpha}A\right)$ is a relative *semi*α-subspace* of an $s^*\alpha$-T_3-*space* (X,τ). We first show that $\left(A, {}_{\tau}^{s^*\alpha}A\right)$ is an $s^*\alpha$-T_1-*space*. Let $x\neq y\in A\subseteq X$. Then there are two *semi*α-open* subsets U and V of (X,τ) containing x and y, respectively, such that $x\notin V$ and $y\notin U$. Now, $G=A\cap U$ and $H=A\cap V$ are two supra *semi*α-open* subsets of $\left(A, {}_{\tau}^{s^*\alpha}A\right)$ containing x and y, respectively, such that $x\notin H$ and $y\notin G$. Thus, $\left(A, {}_{\tau}^{s^*\alpha}A\right)$ is $s^*\alpha$-T_1. Second, we show that $\left(A, {}_{\tau}^{s^*\alpha}A\right)$ is supra *semi*α-regular*. Let H be a supra *semi*α-closed* subset of $\left(A, {}_{\tau}^{s^*\alpha}A\right)$ and $a\in A$ such that $a\notin H$. It follows from Proposition 6.14 that there is a *semi*α-closed* subset F of (X,τ) such that $H=F\cap A$. Since $a\notin F$, then there exist disjoint *semi*α-open* subsets U and V of (X,τ) containing F and a, respectively. Now, $M=U\cap A$ and $N=V\cap A$ are disjoint supra *semi*α-open* subsets of $\left(A, {}_{\tau}^{s^*\alpha}A\right)$ containing H and a, respectively. Thus $\left(A, {}_{\tau}^{s^*\alpha}A\right)$ is supra *semi*α-regular*. Hence, the proof is complete.

Proposition 6.17. Let $g:(X,\tau)\to(Y,\sigma)$ be an injective *semi*α-continuous* mapping. If (Y,σ) is T_i, then (X,τ) is $s^*\alpha$-T_i for $i=0, 1, 2$.

Proof. We only prove the proposition in the case of $i=2$ and the other cases can be made similarly. Let $a\neq b\in X$. Then, it follows from the injectivity of g, that there are $x\neq y\in Y$ such that $x=f(a)$ and $y=f(b)$. Since (Y,σ) is T_2, then there are two disjoint open subsets U and V of (Y,σ) containing a and b, respectively. Now, $g^{-1}(U)$ and $g^{-1}(V)$ are disjoint *semi*α-open* subsets of (X,τ) containing a and b, respectively. Hence, (X,τ) is $s^*\alpha$-T_2, as required. In a similar way, one can prove the following results.

Proposition 6.18. Let $g:(X,\tau)\to(Y,\sigma)$ be a bijective *semi*α-open* mapping. If (X,τ) is T_i, then (Y,σ) is $s^*\alpha$-T_i for $i=0, 1, 2$.

Proposition 6.19. Let $g:(X,\tau)\to(Y,\sigma)$ be an injective *semi*α-continuous* mapping. If (X,τ) is $s^*\alpha$-T_i, then (Y,σ) is $s^*\alpha$-T_i, for $i=0, 1, 2$.

Proposition 6.20. Let $g:(X,\tau)\to(Y,\sigma)$ be a bijective *semi*α-homeomorphism*. If (X,τ) is $s^*\alpha$-T_i then if and only if (Y,σ) is $s^*\alpha$-T_i for $i=0, 1, 2, 3, 4$.

Theorem 6.21. Two sets A and B are *semi*α-open* sets in a topological space (X,τ) if and only if the product $A\times B$ of them is *semi*α-open* in the product topological space $X\times X$.

Proof. Let A and B be two *semi*α-open* sets in a topological space (X,τ). Then $A\subseteq Cl^*\left[\alpha\text{-}Int(A)\right]$ and $B\subseteq Cl^*\left[\alpha\text{-}Int(B)\right]$. So $A\times B\subseteq Cl^*\left[\alpha\text{-}Int(A)\right]\times Cl^*\left[\alpha\text{-}Int(B)\right]=Cl^*\left[\alpha\text{-}Int(A\times B)\right]$. Thus $A\times B$ is a *semi*α-open* set. Conversely, let $A\times B$ be a *semi*α-open* set. Then $A\times B\subseteq Cl^*\left[\alpha\text{-}Int(A\times B)\right]=$ $=Cl^*\left[\alpha\text{-}Int(A)\right]\times Cl^*\left[\alpha\text{-}Int(B)\right]$. So $A\subseteq Cl^*\left[\alpha\text{-}Int(A)\right]$ and $B\subseteq Cl^*\left[\alpha\text{-}Int(B)\right]$. Hence, the proof is complete.

Theorem 6.22. The finite product of $s^*\alpha$-T_i-*spaces* is $s^*\alpha$-T_i-*space* for $i=0, 1, 2$.

Proof. We shall suffice with proof of case $i=2$ which directly contains the case $i=1$. In a similar way, one can

prove the cases $i=0,2$. For simplicity, the proof is given for two topological spaces (X,τ) and (Y,σ).

Let $(X\times Y,T)$ be the product space of (X,τ) and (Y,σ). Suppose that $(x_1,y_1)\neq(x_2,y_2)\in X\times Y$. Then either $x_1\neq x_2$ or $y_1\neq y_2$. Without loss of generality, suppose that $x_1\neq x_2$. Therefore there exist two disjoint *semi*α-open* subsets U and V of (X,τ) containing x_1 and x_2, respectively. It follows from Theorem 6.21 that $U\times Y$ and $V\times Y$ are two *semi*α-open* subsets of $(X\times Y,T)$ containing (x_1,y_1) and (x_2,y_2), respectively, such that $(U\times Y)\cap(V\times Y)=\phi$. Hence, $(X\times Y,T)$ is *s*α-T_2-space*.

Definition 6.23. Let (X,τ) and (Y,σ) be two topological spaces and $(X\times Y,T)$ be their product space such that ξ_1 and ξ_2 are the collections of all *semi*α-open* subsets of (X,τ) and (Y,σ), respectively. Then $\beta=\{G\times H: G\in\xi_1 \text{ and } H\in\xi_2\}$ defines a basis for a supra topology ξ on $X\times Y$. We called $(X\times Y,\xi)$ a *semi*α-finite* product supra space.

Lemma 6.24. Let (X,τ) and (Y,σ) be two topological spaces and $(X\times Y,\xi)$ be their *semi*α-finite* supra space. If E is a supra closed subset of $(X\times Y,\xi)$, then $E=\bigcap_{i\in I}\left[\left(F_i\times Y\right)\cup\left(X\times H_i\right)\right]$, where F_i and H_i are *semi*α-closed* subsets of (X,τ) and (Y,σ), respectively.

Theorem 6.25. The *semi*α-finite* product of *s*α-T_i-spaces* is *s*α-T_i-space* for $i=0,1,2,3$.

Proof. We shall suffice with proof of case $i=3$ which directly contains the case $i=1$. In a similar way, one can prove the cases $i=0,2$. For simplicity, the proof is given for two topological spaces (X,τ) and (Y,σ).

Let $(X\times Y,\xi)$ be the *semi*α-finite* product supra space of (X,τ) and (Y,σ). We first prove that $(X\times Y,\xi)$ is *s*α-T_1*. Suppose that $(x_1,y_1)\neq(x_2,y_2)\in X\times Y$. Then either $x_1\neq x_2$ or $y_1\neq y_2$. Without loss of generality, suppose that $x_1\neq x_2$. Therefore there exist two *semi*α-open* subsets U and V of (X,τ) such that $x_1\in U_1$, $x_2\notin U_1$ and $x_2\in V$, $x_1\notin V_1$. respectively. According to Definition 6.23, $U\times Y$ and $V\times Y$ are two *semi*α-open* subsets of $(X\times Y,\xi)$ containing (x_1,y_1) and (x_2,y_2) such that $(x_1,y_1)\notin V\times Y10$ and $(x_2,y_2)\notin U\times Y$. Hence, $(X\times Y,\xi)$ is *s*α-T_1*. Second, we prove that $(X\times Y,\xi)$ is supra *semi*α-regular*. Suppose that $(x,y)\in X\times Y$ and E is a supra *semi*α-closed* subset of $(X\times Y,\xi)$ such that $(x,y)\notin E=\bigcap_{i\in I}\left[\left(F_i\times Y\right)\cup\left(X\times H_i\right)\right]$, where F_i and H_i are *semi*α-closed* subsets of (X,τ) and (Y,σ), respectively. Then there exists $j\in I$ such that $(x,y)\notin\left[\left(F_j\times Y\right)\cup\left(X\times H_j\right)\right]$. This means that $x\notin F_j$ and $y\notin H_j$. Since (X,τ) and (Y,σ) are *semi*α-regular*, then there exist disjoint *semi*α-open* subsets U and V of (X,τ) containing x and F_j, respectively, and there exist disjoint *semi*α-open* subsets M and N of (Y,σ) containing y and H_j, respectively. Therefore $U\times M$ and $\left[(V\times Y)\cup(X\times N)\right]$ are two supra *semi*α-open* subsets of $(X\times Y,\xi)$ containing (x,y) and $\left[\left(F_j\times Y\right)\cup\left(X\times H_j\right)\right]$, respectively. Obviously, $E\subseteq\left[\left(F_j\times Y\right)\cup\left(X\times H_j\right)\right]$ and $(U\times M)\cap\left[(V\times Y)\cup(X\times N)\right]=\phi$. Thus, $(X\times Y,T)$ is *semi*α-regular*. Hence, the proof is complete.

7. LIMIT POINTS OF A SET WITH RESPECT TO SEMI-STAR-ALPHA-OPEN-SETS

In this section, we introduce and study *semi*α-limit* points of a set. We explore many properties of them and discuss their behavior on the spaces that possess the difference property.

Definition 7.1. A subset A of (X,τ) is said to be a *semi*α-neighbourhood* of $x\in X$ provided that there is a

semi**α-open* set F containing x such that $x \in F \subseteq A$.

Definition 7.2. A point $x \in X$ is said to be a *semi***α-limit* point of a subset A of (X, τ) provided that every *semi***α-neighbourhood* of x contains at least one point of A other than x itself.

The set of all *semi***α-limit* points of A is said to be a *semi***α-derived* set of A and is denoted by $s^*\alpha\text{-}d(A)$.

Proposition 7.3. Let (X, τ) be a topological space and let $A \subseteq B \subseteq X$. Then show that $s^*\alpha\text{-}d(A) \subseteq s^*\alpha\text{-}d(B)$.

Proof. Straightforward.

Corollary 7.4. Let (X, τ) be a topological space and let A and B be subsets of X. Then the following statements are true.

(i) $s^*\alpha\text{-}d(A) \cup s^*\alpha\text{-}d(B) \subseteq s^*\alpha\text{-}d(A \cup B)$.

(ii) $s^*\alpha\text{-}d(A \cap B) \subseteq s^*\alpha\text{-}d(A) \cap s^*\alpha\text{-}d(B)$.

Proposition 7.5. Let A be a subset of (X, τ) and $x \in X$. Then $x \in s^*\alpha\text{-}d(A)$ if and only if $x \in s^*\alpha\text{-}d(A - \{x\})$.

Proof. Necessity: Let $x \in s^*\alpha\text{-}d(A)$. Then for every *semi***α-open* set G containing x, we have $(G - \{x\}) \cap A \neq \phi$. Therefore $(G - \{x\}) \cap (A - \{x\}) \neq \phi$. Thus $x \in s^*\alpha\text{-}d[A - (\{x\})]$.

Sufficiency: It follows from Proposition 7.3.

Theorem 7.6. Let A be a subset of (X, τ). Then the following results hold.

(i) A is a *semi***α-closed* set if and only if $s^*\alpha\text{-}d(A) \subseteq A$.

(ii) $A \cup s^*\alpha\text{-}d(A)$ is a *semi***α-closed* set.

(iii) $s^*\alpha\text{-}Cl(A) = A \cup s^*\alpha\text{-}d(A)$.

Proof. (i) Suppose that A is a *semi***α-closed* set and $x \notin A$. Then A^c is a *semi***α-open* set containing x. In this case $A^c \cap A = \phi$ leads to $x \notin s^*\alpha\text{-}d(A)$. Therefore $s^*\alpha\text{-}d(A) \subseteq A$.

Conversely, let $x \in A^c$ and let $s^*\alpha\text{-}d(A) \subseteq A$. Then $x \notin s^*\alpha\text{-}d(A)$. Therefore there is a *semi***α-open* set G_x containing x such that $(G_x - \{x\}) \cap A = \phi$. Since $x \in A^c$, then $G_x \cap A = \phi$. Now, $G_x \subseteq A^c$. Therefore $A^c = \cup\{G_x : x \in A^c\}$ and hence A^c being the union of *semi***α-open* sets is *semi***α-open*. Thus A is *semi***α-closed*.

(ii) Let $x \notin [A \cup s^*\alpha\text{-}d(A)]$. Then $x \notin A$ and $x \notin s^*\alpha\text{-}d(A)$. Therefore there is a *semi***α-open* set G such that $G \cap A = \phi$. Furthermore, for each $x \in G$, we have $x \notin s^*\alpha\text{-}d(A)$. This means that $G \cap s^*\alpha\text{-}d(A) = \phi$. Consequently, we obtain $G \cap [A \cup s^*\alpha\text{-}d(A)] = \phi$. This implies that $x \notin s^*\alpha\text{-}d[A \cup s^*\alpha\text{-}d(A)]$. Hence $s^*\alpha\text{-}d[A \cup s^*\alpha\text{-}d(A)] \subseteq [A \cup s^*\alpha\text{-}d(A)]$. By (i), $A \cup s^*\alpha\text{-}d(A)$ is a *semi***α-closed* set, as required.

(iii) Since $A \subseteq s^*\alpha\text{-}Cl(A)$ and $s^*\alpha\text{-}d(A) \subseteq s^*\alpha\text{-}d[s^*\alpha\text{-}Cl(A)] \subseteq s^*\alpha\text{-}Cl(A)$, then clearly we obtain

$A \cup s^*\alpha\text{-}d(A) \subseteq s^*\alpha\text{-}Cl(A)$. Since $A \cup s^*\alpha\text{-}d(A)$ is a *semi*α-closed* set containing A and $s^*\alpha\text{-}Cl(A)$ is the smallest *semi*α-closed* set containing A, then $s^*\alpha\text{-}Cl(A) \subseteq A \cup s^*\alpha\text{-}d(A)$. Therefore we conclude that $s^*\alpha\text{-}Cl(A) = A \cup s^*\alpha\text{-}d(A)$.

Corollary 7.7. If A is a *semi*α-closed* subset of a topological space (X, τ), then $s^*\alpha\text{-}d(A)$, $s^*\alpha\text{-}d\left(s^*\alpha\text{-}d(A)\right)$, $s^*\alpha\text{-}d\left(s^*\alpha\text{-}d\left(s^*\alpha\text{-}d(A)\right)\right)$, are *semi*α-closed* sets.

Definition 7.8. A mapping $g:(X, \tau) \to (Y, \sigma)$ is said to be:

(i) *semi*α-continuous* if $g^{-1}(H)$ is a *semi*α-open* set in X for every *semi*α-open* set in Y.

(ii) *semi*α-open* $\left(resp.\ semi^*\alpha\text{-}closed\right)$ if $g(H)$ is a *semi*α-open* $\left(resp.\ semi^*\alpha\text{-}closed\right)$ set in Y for every *semi*α-open* $\left(resp.\ semi^*\alpha\text{-}closed\right)$ set in X.

(iii) *semi*α-homeomorphism* if it is bijective, *semi*α-continuous* and *semi*α-open.*

Theorem 7.9. If $g:(X, \tau) \to (Y, \sigma)$ is a *semi*α-homeomorphism* mapping, then $g\left(semi^*\alpha\text{-}d(A)\right) = semi^*\alpha\text{-}d\left(g(A)\right)$ for each $A \subseteq X$.

Proof. Let $a \notin s^*\alpha\text{-}d\left(g(A)\right)$. Then there is a *semi*α-open* set H containing a such that $(H - \{a\}) \cap g(A) = \phi$. So $g^{-1}\left[(H - \{a\}) \cap g(A)\right] = g^{-1}(\phi) = \phi$. This implies that $\left[g^{-1}(H) - g^{-1}(a)\right] \cap A = \phi$. Thus $g^{-1}(a) \notin s^*\alpha\text{-}d(A)$. Since g is bijective, then $a \notin g\left(s^*\alpha\text{-}d(A)\right)$. Therefore $g\left(s^*\alpha\text{-}d(A)\right) \subseteq s^*\alpha\text{-}d\left(g(A)\right)$. By reversing the preceding steps, we find that $s^*\alpha\text{-}d\left(g(A)\right) \subseteq g\left(s^*\alpha\text{-}d(A)\right)$. Hence, the proof is complete.

Definition 7.10. For a nonempty set X, a sub collection Λ of 2^X is said to have the difference property provided that $G \in \Lambda$ implies that $G - \{x\} \in \Lambda$.

Theorem 7.11. If (X, τ) has the difference property for the collection of *semi*α-open* sets, then the following properties hold for $A \subseteq X$.

(i) $s^*\alpha\text{-}d\left[s^*\alpha\text{-}d(A)\right] \subseteq s^*\alpha\text{-}d(A)$.

(ii) $s^*\alpha\text{-}Cl\left[s^*\alpha\text{-}d(A)\right] = s^*\alpha\text{-}d(A) = s^*\alpha\text{-}d\left[s^*\alpha\text{-}Cl(A)\right]$.

(iii) $s^*\alpha\text{-}d(A) = \phi$ if A is finite.

Proof. (i) Let $x \notin s^*\alpha\text{-}d(A)$. Then there is a *semi*α-open* set G containing x such that $(G - \{x\}) \cap A = \phi$. Since (X, τ) has the difference property for the collection of *semi*α-open* sets, then $G - \{x\}$ is a *semi*α-open* set. Therefore $(G - \{x\}) \cap s^*\alpha\text{-}d(A) = \phi$. Since $x \notin s^*\alpha\text{-}d(A)$, then $G \cap s^*\alpha\text{-}d\left[s^*\alpha\text{-}d(A)\right] = \phi$. Thus $x \notin s^*\alpha\text{-}d\left[s^*\alpha\text{-}d(A)\right]$. So, $s^*\alpha\text{-}d\left[s^*\alpha\text{-}d(A)\right] \subseteq s^*\alpha\text{-}d(A)$.

(ii) Since $s^*\alpha\text{-}d\left[s^*\alpha\text{-}d(A)\right] \subseteq s^*\alpha\text{-}d(A)$, then it follows from Theorem 7.6 that $s^*\alpha\text{-}d(A)$ is a *semi*α-closed* set. Therefore $s^*\alpha\text{-}Cl\left[s^*\alpha\text{-}d(A)\right] = s^*\alpha\text{-}d(A)$. Also, $s^*\alpha\text{-}d(A) \subseteq s^*\alpha\text{-}d\left[s^*\alpha\text{-}Cl(A)\right]$ because $A \subseteq s^*\alpha\text{-}Cl(A)$. On the other

hand, let $x \notin s^*\alpha\text{-}d(A)$. Then it follows from (i) that $(G-\{x\})\cap A=\phi$ and $(G-\{x\})\cap s^*\alpha\text{-}d(A)=\phi$. This means that $(G-\{x\})\cap s^*\alpha\text{-}Cl(A)=\phi$. Therefore $x \notin s^*\alpha\text{-}d\left[s^*\alpha\text{-}Cl(A)\right]$. Thus $s^*\alpha\text{-}d\left[s^*\alpha\text{-}Cl(A)\right]\subseteq s^*\alpha\text{-}d(A)$. Hence $s^*\alpha\text{-}d\left[s^*\alpha\text{-}Cl(A)\right]=s^*\alpha\text{-}d(A)$. Consequently, the desired result is proved.

(iii) Let A be a finite subset of X. Suppose that there exists an element $x\in X$ such that $x\in s^*\alpha\text{-}d(A)$. Then for every *semi*α-open* set G containing x, we have $(G-\{x\})\cap A\neq\phi$. Therefore for every $y\in A$ such that $y\neq x$, we have $G-\{x, y\}$ is a *semi*α-open* set. Thus $G-(A\cup\{x\})$ is a *semi*α-open* set such that $\left[G-(A\cup\{x\})\right]\cap A=\phi$. This implies that $x\notin s^*\alpha\text{-}d(A)$. But this is a contradiction. Hence, it must be that $s^*\alpha\text{-}d(A)=\phi$.

8. CONCLUSIONS

Topological vector spaces are a fundamental notion and play an important role in various advanced branches of mathematics like fixed point theory, operator theory, etc. We introduce and study a new class of space, namely *semi*α-irresolute* topological vector spaces via *semi*α-open* sets. We investigate several new properties and characterizations of *semi*α-irresolute* topological vector spaces. We present several properties and characterizations of contra semi$^*\alpha$-continuous functions in topological spaces. We use contra semi$^*\alpha$-continuity to introduce contra semi$^*\alpha$-topological vector spaces. We explore some more basic features of contra semi$^*\alpha$-topological vector spaces and interprets their relationships with some well-known existing spaces. We utilize *semi*α-open* sets to introduce the concepts of *semi*α-regular*, *semi*α-normal* and $s^*\alpha\text{-}T_i$*-spaces* $(i=0,1,2,3,4)$. We give some characterizations for each one of them and elucidate the relationships among themselves. Furthermore, we introduce and study *semi*α-limit* points of a set. We explore many properties of them and discuss their behavior on the spaces that possess the difference property.

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