A Novel Weighted Fuzzy Harmonic Mean Aggregation Operator on Fuzzy Graphs for Multi-Criteria Decision Making and its Application in AI-Based Recommendation Systems

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ABSTRACT

Multi-criteria decision making (MCDM) under uncertainty often uses aggregation operators to combine fuzzy criteria evaluations into a single score. However, commonly used weighted arithmetic and geometric operators tend to over-reward alternatives that perform well on some criteria while masking weak performance on others undesirable in risk-sensitive applications like recommendation systems. This paper proposes a Weighted Fuzzy Harmonic Mean Aggregation Operator (WFHMAO) defined over a fuzzy decision graph, where alternatives are vertices and criteria-wise satisfaction degrees are fuzzy edge memberships. Its harmonic mean structure penalizes low membership values more strongly than arithmetic or geometric aggregation, making it well-suited for cases where a single poor criterion should meaningfully lower an alternative's score. We formally establish idempotency, boundedness, monotonicity, and commutativity of the operator, and embed it as a fusion layer in a hybrid AI recommendation framework combining content-based similarity, collaborative-filtering scores, and popularity signals. A case study on online course recommendation, compared against weighted arithmetic mean (WAM) and weighted geometric mean (WGM), shows that WFHMAO produces more conservative, discrimination-sensitive rankings with reduced influence of criterion-specific outliers.

Keywords: Fuzzy Graph; Harmonic Mean, Aggregation Operator, Multi-Criteria Decision Making, Recommendation System, Artificial Intelligence, Fuzzy Sets.

INTRODUCTION

Since Zadeh's introduction of fuzzy set theory, fuzzy logic has become a central tool for modelling imprecision and partial truth in decision-making problems where information is vague, incomplete, or subjectively assessed. Rosenfeld's extension of fuzzy sets to graph-theoretic structures, giving rise to fuzzy graphs, provided a natural way to represent relationships between entities whose strength of association is itself uncertain. Fuzzy graphs have since been applied to network analysis, clustering, transportation problems and, increasingly, to decision support systems where alternatives and criteria can be modelled as vertices and fuzzy-weighted edges.

Recommendation systems illustrate this challenge concretely. A modern recommender rarely relies on a single signal: it typically blends content-based similarity, collaborative-filtering scores, price or affordability, popularity, and trust or reputation indicators, each of which is uncertain, noisy, or only partially known for any given user–item pair. Treating these heterogeneous, imprecise signals as fuzzy memberships and combining them through a fuzzy-graph-structured decision model provides a principled way to reason about the overall desirability of a candidate item, provided the aggregation operator itself behaves in a manner consistent with how a careful human decision-maker would weigh strengths against weaknesses.

A recurring challenge in fuzzy multi-criteria decision making (MCDM) is the choice of aggregation operator used to combine per-criterion fuzzy scores into an overall evaluation. The weighted arithmetic mean (WAM) is simple but compensatory: a very poor score on one criterion can be offset by strong scores on others. The weighted geometric mean (WGM) is less compensatory but still tolerates moderate weaknesses. In many real applications including product recommendation, course recommendation, vendor selection, and healthcare triage a poor score on even a single important criterion (for example, safety, affordability, or relevance) should disproportionately reduce the overall desirability of an alternative. The harmonic mean, well known in classical statistics for being dominated by the smallest values in a data set, is a natural candidate for such conservative aggregation, yet it has received comparatively little attention in the fuzzy-graph MCDM literature, and even less in combination with AI-based recommendation pipelines.

This paper addresses that gap. We define a fuzzy decision graph in which vertices represent decision alternatives and (criterion-indexed) edge memberships represent the degree to which an alternative satisfies each criterion. Over this structure we define a Weighted Fuzzy Harmonic Mean Aggregation Operator (WFHMAO) and prove its core algebraic properties. We then show how WFHMAO can be embedded as an aggregation layer inside a hybrid AI-based recommendation architecture, where criterion scores are themselves produced by content-based and collaborative-filtering sub-models, and operator weights are learned from historical interaction data.

The main contributions of this paper are:

- A formal definition of a fuzzy decision graph and a Weighted Fuzzy Harmonic Mean Aggregation Operator (WFHMAO) constructed over it.
- Proofs of idempotency, boundedness, monotonicity and commutativity for the proposed operator.
- A hybrid AI-based recommendation framework in which WFHMAO serves as the fusion layer combining multiple heterogeneous scoring signals.
- An illustrative numerical case study and a comparative analysis against weighted arithmetic and geometric mean operators, evaluating ranking behaviour and sensitivity to criterion-specific outliers.

The remainder of the paper is organized as follows. Section 2 reviews related work on fuzzy graphs, aggregation operators and AI-based recommendation. Section 3 presents preliminaries. Section 4 introduces the proposed operator and its properties. Section 5 describes the AI-based recommendation framework. Section 6 presents the case study, Section 7 the comparative analysis, Section 8 discusses the results, and Section 9 concludes the paper with directions for future work.

Related Work

Fuzzy set theory, introduced by Zadeh, laid the foundation for representing partial membership and has since been extended in numerous directions, including intuitionistic fuzzy sets, hesitant fuzzy sets and fuzzy graphs. Rosenfeld formalized fuzzy graphs by attaching membership degrees to vertices and edges, enabling graph-theoretic reasoning under uncertainty; subsequent work extended this to bipolar, interval-valued and Pythagorean fuzzy graphs, broadening the range of relationships that can be modelled.

Aggregation operators are central to fuzzy MCDM. Yager's ordered weighted averaging (OWA) operator generalized the weighted mean by allowing weights to be assigned to ordered positions rather than fixed criteria, enabling a spectrum of behaviour between the maximum and minimum operators. Xu and co-authors, among others, studied harmonic-mean-based aggregation operators for intuitionistic and interval-valued fuzzy information, showing that harmonic aggregation tends to produce more conservative rankings than arithmetic or geometric alternatives. Saaty's Analytic Hierarchy Process (AHP) remains a widely used method for deriving criterion weights from pairwise comparisons, while entropy-weight methods derive weights directly from the dispersion of the decision data, reducing subjectivity.

Harmonic-mean-based aggregation has a long history in classical statistics for instance in computing average rates and ratios, and in information retrieval, where the F-measure is itself the harmonic mean of precision and recall precisely because it penalizes an imbalance between the two more strongly than an arithmetic average would. This same penalizing property motivated early work on fuzzy and intuitionistic harmonic aggregation operators, which showed analytically that harmonic-mean-based operators sit below their geometric- and arithmetic-mean counterparts in value ($H \leq G \leq A$), and are therefore inherently less compensatory. More recently, Shit, Ghorai, Xin and Gulzar developed weighted, ordered-weighted and hybrid harmonic mean operators for trapezoidal picture fuzzy numbers and demonstrated, through a worked numerical example, that harmonic aggregation is markedly less sensitive to a single abnormally high data point than the corresponding arithmetic mean, reinforcing the motivation for a harmonic-based operator in settings with heterogeneous, possibly unbalanced criterion scores. In a related direction, Irvanizam and Zahara combined harmonic and arithmetic mean operators with a single-valued trapezoidal neutrosophic ranking method for healthcare service quality evaluation, again showing the practical value of harmonic aggregation in multi-criteria service-quality settings.

On the graph-theoretic side, fuzzy graphs continue to be actively applied to decision problems: Bhattacharya and Pal used a fuzzy-graph-theoretic approach to solve a facility-location problem in the Indian banking sector, while Ahmad and Sabir proposed multicriteria decision-making indices based on degree- and distance-based measures of fuzzy graphs. These works confirm that fuzzy-graph structures remain an active and practically relevant modelling choice for MCDM, but neither combines a fuzzy decision graph with a harmonic-mean aggregation operator, nor embeds the resulting operator within an AI-based recommendation pipeline — the specific combination addressed in this paper.

On the recommendation-systems side, classical approaches are grouped into content-based filtering, collaborative filtering and hybrid methods that combine multiple signals to mitigate the cold-start and sparsity limitations of any single technique; Ko, Lee, Park and Choi provide a recent survey of recommendation models, techniques and application fields spanning these categories. Karn et al. proposed a customer-centric hybrid e-commerce recommender that integrates sentiment analysis with collaborative signals, reporting accuracy gains over single-method baselines, while Deng, Chen, Wang, Ye and Wang recently proposed a fuzzy neural collaborative filtering model that embeds fuzzy membership reasoning directly into a neural collaborative filtering architecture. Fuzzy-logic-based recommenders in general aim to better model the inherent imprecision in user preferences and item attributes, typically using fuzzy rule bases or fuzzy similarity measures, but as in the fuzzy-graph MCDM literature above existing hybrid fuzzy recommenders predominantly rely on arithmetic or weighted-sum fusion of component scores rather than a conservative, harmonic mean based fusion layer. To the best of our knowledge, the combination of a fuzzy graph structured decision model with a harmonic-mean-based aggregation operator, embedded specifically as a fusion layer within an AI recommendation pipeline, has not been systematically formalized this is the gap the present work addresses.

Preliminaries

Definition 3.1 (Fuzzy Set). A fuzzy set A on a universe X is characterized by a membership function

$\mu A : X \rightarrow [0, 1]$, where (x) denotes the degree to which x belongs to A.

Definition 3.2 (Fuzzy Graph). A fuzzy graph is a pair $G = (\sigma, \mu)$ on a crisp vertex set V, where $\sigma :$

$V \rightarrow [0, 1]$ is the fuzzy vertex membership and $\mu : V \times V \rightarrow [0, 1]$ is the fuzzy edge membership satisfying $(u, v) \leq (u) \wedge (v)$ for all $u, v \in V$, with $\wedge$ denoting the minimum operator.

Definition 3.3 (Multi-Criteria Decision Problem). Let $A = \{A_1, A_2, \ldots, Am\}$ be a finite set of alternatives and $C = \{C_1, C_2, \ldots, Cn\}$ a finite set of criteria, each with an associated weight $wj \in$

$(0, 1]$, $\sum j\ wj = 1$. Each alternative criterion pair (Aj, Cj) is assigned a fuzzy satisfaction degree $\mu ij \in$

$(0, 1]$, representing the extent to which alternative Ai satisfies criterion Cj. The decision maker seeks an aggregation operator F such that $si = (\mu i1, \mu i2, \ldots \mu in)$ yields an overall score si for each alternative, inducing a ranking over A.

The harmonic mean satisfies $H \leq G \leq A$, where G and A are the geometric and arithmetic means respectively, with equality if and only if all xi are equal. This inequality is the formal basis for the conservative, outlier-sensitive behaviour that motivates its use in risk-aware decision aggregation.

Proposed Weighted Fuzzy Harmonic Mean Aggregation Operator (WFHMAO) Fuzzy Decision Graph

We construct a fuzzy decision graph $G = (V, E, \mu)$ as follows. The vertex set $V = A \cup C$ consists of alternative-vertices $A_1, A_2, \ldots, Am$ and criterion-vertices $C_1, C_2, \ldots, Cn$. For every alternative Ai and criterion Cj, an edge (Ai, Cj) is included with fuzzy membership $(Ai, Cj) = \mu ij \in (0, 1]$, obtained from a fuzzification module that converts raw (possibly heterogeneous) attribute values into normalized membership degrees. Each criterion-vertex Cj additionally carries a crisp importance weight wj, $\sum j\ wj = 1$, representing the relative significance of that criterion in the decision problem.

Definition of WFHMAO

Definition 4.1. For an alternative Ai with criterion-wise fuzzy memberships $\mu i1, \mu i2, \ldots \mu in$ along its incident edges in G, and criterion weights $w1,, \ldots, w1$, $(\Sigma wj = 1$, wj > 0), the Weighted Fuzzy Harmonic Mean Aggregation Operator is defined as:

$$WFHMAO(\mu_{i1}, \mu_{i2}, \ldots \mu_{in}) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{\mu_{ij}}}$$

The resulting score $si = WFH(\mu i1, \mu i2, \ldots \mu in) \in (0, 1]$ serves as the overall fuzzy desirability of alternative Ai, and alternatives are ranked in decreasing order of si.

Properties of WFHMAO

Theorem 4.1 (Idempotency). If $(\mu_{i1}, \mu_{i2}, \ldots \mu_{in}) = \mu$, then WFHMAO$(\mu_{i1}, \mu_{i2}, \ldots \mu_{in}) = \mu$.

Proof. Substituting $\mu_{ij} = \mu$ for all j gives $WFHMAO = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{\mu}} = \frac{1}{\frac{1}{\mu}\sum_{j=1}^{n} w_j} = \frac{1}{\frac{1}{\mu}} = \mu$, since $\sum_{j=1}^{n} w_j = 1$. ■

Theorem4.2 (Boundedness). $m\{\mu i1, \mu i2, \ldots \mu in\} \leq WFHMAO(\mu i1, \mu i2, \ldots \mu in) \leq max\{\mu i1, \mu i2, \ldots \mu in\}$.

Proof. Let $\mu_{min} = min\{\mu_{ij}\}$ and $\mu_{max} = max\{\mu_{ij}\}$. Since $\mu_{min} \leq \mu_{ij} \leq \mu_{max}$ for all j, we have $\frac{1}{\mu_{max}} \leq \frac{1}{\mu_{ij}} \leq \frac{1}{\mu_{min}}$. Multiplying by $w_j > 0$ and summing over j: $\frac{1}{\mu_{max}} = \sum_{j=1}^{n} \frac{w_j}{\mu_{max}} \leq \sum_{j=1}^{n} \frac{w_j}{\mu_{ij}} \leq \sum_{j=1}^{n} \frac{w_j}{\mu_{min}}$ (using $\sum_{j=1}^{n} w_j = 1$). Taking reciprocals reverses the inequalities, giving $\mu_{min} \leq \frac{1}{\sum_{j=1}^{n} \frac{w_j}{\mu_{ij}}} \leq \mu_{max}$, i.e. $\mu_{min} \leq WFHMAO \leq \mu_{max}$. ■

Theorem 4.3 (Monotonicity). If $\mu_{ij} \leq \mu'_{ij}$ for every criterion j (with the same weights w_j), then

$WFHM(\mu_{i1}, \mu_{i2}, \ldots \mu_{in}) \leq WFHMAO(\mu'_{i1}, \mu'_{i2} \ldots \mu'_{in})$.

Proof. Since $t \mapsto 1/t$ is strictly decreasing on $(0,1]$, $\mu_{ij} \leq \mu'_{ij}$ implies $\frac{1}{\mu_{ij}} > \frac{1}{\mu'_{ij}}$ for each j. Multiplying by $w_j > 0$ and summing, $\sum_{j=1}^{n} \frac{w_j}{\mu_{ij}} \geq \sum_{j=1}^{n} \frac{w_j}{\mu'_{ij}}$. Taking reciprocals (again a decreasing transformation on positive reals) reverses the inequality: $\frac{1}{\sum_{j=1}^{n} \frac{w_j}{\mu_{ij}}} \leq \frac{1}{\sum_{j=1}^{n} \frac{w_j}{\mu'_{ij}}}$, i.e. $WFHMAO(\mu_i) \leq$

$WFHMAO(\mu'_j)$. ■

Theorem 4.4 (Commutativity / Symmetry). For any permutation π of $\{1, \dots, n\}$ applied jointly to the memberships and their matching weights, $WFHMAO(\mu_{i\pi(1)}, \mu_{i\pi(2)}, \dots, \mu_{i\pi(n)}) = WFHMAO((\mu_{i1}, \mu_{i2}, \dots \mu_{in})$.

Proof. Addition in $\sum_{j=1}^{n} \frac{w_j}{\mu_{ij}}$ is commutative, so reordering the (weight, membership) pairs does not change the value of the sum, and hence does not change its reciprocal. ■

Remark 4.1. Because WFHMAO involves division by μ_{ij}, criterion memberships are required to be strictly positive ($\mu_{ij} \in (0, 1]$); in practice a small $\varepsilon > 0$ $(e.g.\ \varepsilon = 0.01)$ is substituted for any raw score that fuzzifies to exactly zero, so that an alternative failing one criterion is heavily penalized rather than assigned an undefined or zero-division score.

Remark 4.2 (Relation to the Inequality of Means). Since the classical inequality $(x_1, x_2 \dots, x_n) \leq$

$(x_1, x_2 \dots, x_n) \leq (x_1, x_2 \dots, x_n)$ holds for any positive reals with equality only when all xi coincide, the weighted fuzzy analogue WFHMAO$((\mu_{i1}, \mu_{i2}, \dots \mu_{in}) \leq WGM((\mu_{i1}, \mu_{i2}, \dots \mu_{in}) \leq$

$(\mu_{i1}, \mu_{i2}, \dots \mu_{in})$ also holds for the corresponding weighted geometric and arithmetic operators. This ordering formally guarantees that WFHMAO is never more optimistic than WAM or WGM for the same input data and weights, which is the theoretical basis for the conservative ranking behaviour observed in the case study of Section 6.

A Minimal Worked Example

To make Definition 4.1 concrete before the full case study, consider two criteria with equal weights

$w_1 = w_2 = 0.5$ and an alternative with $\mu_1 = 0.9$ and $\mu_2 = 0.3$. Then WAM = 0.5(0.9) + 0.5(0.3) = 0.60, WGM = (0.9 × 0.3)^0.5 ≈ 0.520, while $WFHMAO$ = 1 / (0.5/0.9 + 0.5/0.3) = 1 / (0.556 + 1.667) = 1 / 2.222 ≈ 0.450. The weak criterion (μ_1 = 0.3) pulls the harmonic-mean score noticeably below both the arithmetic and geometric scores, illustrating Remark 4.2 in miniature and previewing the behaviour that recurs at larger scale in Section 6.

PROPOSED AI-BASED RECOMMENDATION FRAMEWORK USING WFHMAO

Architecture Overview

The proposed framework integrates WFHMAO as a fusion layer within a hybrid AI recommendation pipeline consisting of five modules, described below.

- Data and Feature Extraction Module: collects user item interaction history, item attributes and contextual signals.
- Fuzzification Module: converts heterogeneous raw scores (numeric ratings, similarity scores, price, popularity counts, etc.) into normalized fuzzy membership degrees in (0, 1] using min–max normalization followed by a sigmoidal or trapezoidal membership function.
- Fuzzy Decision Graph Construction Module: builds the fuzzy decision graph $G = (V, E, \mu)$ described in Section 4.1, with one alternative-vertex per candidate item and one criterion-vertex per scoring signal (e.g., content-based similarity, collaborative-filtering score, price affordability, popularity).
- WFHMAO Aggregation Layer: applies the proposed operator to compute an overall fuzzy desirability score si for every candidate item, using weights wj learned or elicited as described in Section 5.2.
- AI Ranking and Learning Module: ranks candidate items by si to produce the final top$-K$ recommendation list, and periodically updates criterion weights wj using feedback signals (clicks, purchases, ratings) from users.

Weight Determination

Criterion weights wj can be obtained in two complementary ways. First, an entropy-weight method computes wj directly from the dispersion of criterion scores across the candidate set, assigning higher weight to criteria that are more discriminative for the current user session. Second, weights may be learned by gradient-based optimization: treating wj is trainable parameters (subject to $wj > 0$, $\sum_{j=1}^{n} wj = 1$ via a softmax reparameterization), a ranking loss (e.g., pairwise logistic loss over observed click/no-click pairs) is minimized so that WFHMAO scores align with observed user preferences over time.

Algorithm

1. Collect candidate item set for the active user from the retrieval stage (content-based and collaborative-filtering candidate generators).
2. For each candidate item, compute raw criterion scores: content-similarity score, collaborative-filtering score, affordability score, popularity/trust score.
3. Fuzzify each raw score into $\mu_{ij} \in (0, 1]$ via the Fuzzification Module; replace any zero score with ε to avoid division errors.
4. Construct/update the fuzzy decision graph G with alternative and criterion vertices and edge memberships μ_{ij}.
5. Determine criterion weights w_j via the entropy method (cold-start / new session) or the learned weights (returning user with sufficient history).
6. Compute $si = WFHMAO((\mu i1, \mu i2, \ldots \mu in) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{\mu_{ij}}}$ for every candidate item i.
7. Rank candidates in decreasing order of s_i and return the top$-K$ items as the recommendation list.
8. Log user feedback on the presented list and use it to update w_j for subsequent sessions.

ILLUSTRATIVE CASE STUDY: FUZZY HARMONIC-MEAN-BASED ONLINE COURSE RECOMMENDATION

To illustrate the proposed operator, consider an online learning platform that must rank five candidate courses (A1–A5) for a learner, evaluated against four criteria: C_1 − topical relevance (content-based similarity to the learner's stated interests), C_2 −peer rating quality, C_3 −price affordability, and C_4 − instructor reputation. Raw scores are normalized and fuzzified to memberships in (0, 1], shown in Table 1. Criterion weights, obtained via the entropy-weight method for this session, are $w_1 = 0.35$ (relevance), $w_2 = 0.30$ (rating), $w_3 = 0.20$ (affordability), $w_4 = 0.15$ (reputation).

Fuzzification of Raw Scores

Raw, heterogeneous attribute values are first mapped to memberships in (0, 1] by the Fuzzification Module (Section 5.1). For criteria that are naturally already scaled (such as a $0 - 1$ content-similarity score or a normalized rating), a linear min–max mapping is used directly. For the affordability criterion C_3, where the raw attribute is course price in currency units, a decreasing trapezoidal membership function is applied: prices at or below a learner-specific comfortable threshold pmin map to $\mu = 1$, prices at or above an unaffordable threshold pmax map to $\mu \approx 0.05$ (using ε in place of an exact zero, per Remark 4.1), and prices in between are interpolated linearly, , $\mu(price) = 1 - (price - pmin)$ For example, with p_{min} = ₹500 and p_{man}= ₹2500, a course priced at ₹1700 yields $p_{max} - p_{min}$

$\mu = 1 - (1700 - 500)/(2500 - 500) = 1 - 0.60 = 0.40$, matching the C_3 value used for course A1 in $Table$ 1.

Course	Relevance	Rating	Affordability	Reputation
A1	0.9	0.85	0.4	0.8
A2	0.75	0.78	0.7	0.72
A3	0.95	0.55	0.85	0.6
A4	0.6	0.65	0.9	0.55
A5	0.8	0.8	0.75	0.78

Table 1. Fuzzified criterion memberships μ_{ij} for five candidate courses.

Applying Definition 4.1, the WFHMAO score for course A1 is computed as:

$$s(A1) = \frac{1}{\frac{0.35}{0.90} + \frac{0.30}{0.85} + \frac{0.20}{0.40} + \frac{0.15}{0.80}}$$

$$s(A1) = \frac{1}{0.389 + 0.353 + 0.500 + 0.188} = \frac{1}{1.429} \approx 0.700$$

Repeating this computation for all five courses gives the WFHMAO scores and resulting ranking shown in Table 2.

Course	WFHMAO Score	Rank
A1	0.7	4
A2	0.743	2
A3	0.715	3
A4	0.649	5
A5	0.787	1

Table 2. WFHMAO scores and ranking of candidate courses.

The ranking $A5 \succ A2 \succ A3 \succ A1 \succ A4$ reflects the harmonic mean's sensitivity to weak criteria: although A1 scores very highly on relevance (0.90), its low affordability (0.40) pulls its overall WFHMAO score down more sharply than a compensatory operator would allow, dropping it below A2 and A3 — courses that perform more consistently, if less spectacularly, across all four criteria.

COMPARATIVE ANALYSIS WITH EXISTING AGGREGATION OPERATORS

To evaluate the discriminative behaviour of WFHMAO, we recompute the aggregate scores for the same data (Table 1, same weights) using the weighted arithmetic mean (WAM) $s_i = \sum_j w_j \mu_{ij}$ and the weighted geometric mean (WGM) $s_i = (\mu_{ij})$^w_j. Table 3 summarizes the resulting scores and rankings for all three operators.

Course	WA M Score	WA M Rank	WG M Score	WG M Rank	WFHMA O Score	WFHMA O Rank
A1	0.77	2	0.739	3	0.7	4
A2	0.745	4	0.744	2	0.743	2
A3	0.758	3	0.736	4	0.715	3
A4	0.668	5	0.658	5	0.649	5
A5	0.787	1	0.787	1	0.787	1

Table 3. Comparison of WAM, WGM and WFHMAO scores and rankings for the same criterion data and weights.

Two observations are notable. First, all three operators agree that A5 is the top recommendation and A4 the weakest, indicating stability for alternatives that are either consistently strong or consistently weak across criteria. Second, the operators disagree on the middle of the ranking: WAM ranks A1 second on the strength of its high relevance score alone, whereas WFHMAO penalizes A1's low affordability (0.40) far more heavily, dropping it to fourth place, below both A2 and A3. This confirms the theoretical expectation, following from $H \leq G \leq A$ (Section 3), that WFHMAO produces the most conservative and most outlier-sensitive ranking of the three operators.

A brief sensitivity check further illustrates this behaviour: reducing A2's affordability score from 0.70 to 0.30 (simulating a price increase, all other scores unchanged) drops A2 by only one rank position under WAM (from 4th to 5th) and by one position under WGM, but by three rank positions under WFHMAO (from 2nd to 5th), as summarized in Table 4.

Operator	A2 Score (original)	A2 Rank (original)	A2 Score ($C3$ =0.30)	A2 Rank (revised)	Rank Drop
WAM	0.745	4	0.665	5	1
WGM	0.744	2	0.699	3	1
WFHMAO	0.743	2	0.579	5	3

Table 4. Sensitivity of A2's score and rank to a deterioration in a single criterion (affordability), under each aggregation operator.

This confirms that WFHMAO responds far more strongly to deterioration on any single criterion than WAM or WGM do a desirable property in applications such as course or product recommendation, where a critical weakness (e.g., unaffordable price) should not be masked by strong performance elsewhere. Computationally, WFHMAO has the same (mn) time complexity as WAM and WGM for m alternatives and n criteria, so this added sensitivity comes at no additional computational cost.

RESULTS AND DISCUSSION

The theoretical results of Section 4 establish that WFHMAO is a well-behaved aggregation operator idempotent, bounded between the minimum and maximum criterion memberships, monotonic in each criterion, and independent of criterion ordering making it a mathematically valid alternative to arithmetic and geometric fuzzy aggregation. The case study and comparative analysis in Sections 6–7 show that this theoretical conservatism translates into practically meaningful ranking differences: alternatives with an unevenly distributed criterion profile (very strong on some criteria, weak on others) are ranked lower under WFHMAO than under WAM or WGM, while alternatives with balanced, consistently good performance are favoured. This behaviour aligns naturally with recommendation contexts where a single disqualifying weakness (unaffordable price, poor reviews, low relevance) should not be compensated away by unrelated strengths.

From a systems perspective, embedding WFHMAO as the fusion layer of a hybrid AI recommender (Section 5) allows the framework to combine heterogeneous scoring signals which may come from entirely different sub-models such as embedding-based similarity and collaborative-filtering matrix factorization within a single, interpretable, and theoretically justified aggregation step, while still permitting the underlying criterion weights to be learned or adapted from user feedback over time.

From a scalability standpoint, the O(mn) complexity noted in Section 7 means WFHMAO can be evaluated for large candidate sets (as generated by a typical two-stage retrieve-then-rank recommender architecture) without materially affecting online serving latency, since m is usually bounded by the size of the shortlist returned by the retrieval stage (a few hundred items) rather than the full catalogue.

Limitations. Two limitations of the present work should be acknowledged. First, WFHMAO's conservatism, while desirable when a single weak criterion should be disqualifying, may be too punitive in settings where compensatory trade-offs between criteria are explicitly acceptable to the decision-maker; in such cases a tunable hybrid operator (Section 9) may be preferable. Second, the illustrative case study in Section 6 uses a small, manually constructed data set to keep the computation transparent and verifiable by hand; the framework's practical benefits over WAM/WGM fusion should be confirmed on larger real-world interaction logs, ideally through online A/B testing rather than offline scoring alone.

CONCLUSION AND FUTURE SCOPE

This paper proposed a Weighted Fuzzy Harmonic Mean Aggregation Operator (WFHMAO) defined over a fuzzy decision graph, proved its idempotency, boundedness, monotonicity and commutativity, and demonstrated its use as a fusion layer within a hybrid AI-based recommendation framework. A worked case study and comparative analysis against weighted arithmetic and geometric mean operators showed that WFHMAO produces more conservative, outlier-sensitive rankings that better reflect decision contexts in which a single weak criterion should meaningfully penalize an alternative.

Future work includes: (i) extending WFHMAO to intuitionistic and Pythagorean fuzzy graph settings to capture both membership and non-membership degrees; (ii) empirically validating the proposed recommendation framework on large-scale real-world interaction datasets (e.g., e-commerce or e-learning platforms) with online A/B testing; (iii) developing an adaptive weight-learning scheme that combines entropy-based and gradient-based weight estimation; and (iv) exploring hybrid harmonic–geometric operators that allow a tunable compensation parameter between conservative and compensatory aggregation behaviour.

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